Module 1: Polynomial, Rational, and Radical Relationships

**Unit Description:** In this module, students draw on their foundation of the analogies between polynomial arithmetic and base ten computation, focusing on properties of operations, particularly the distributive property (A-APR.1, ASSE.2). Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers (A-APR.1, A-APR.6). Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations (A-APR.3). The role of factoring, as both an aid to the algebra and to the graphing of polynomials, is explored (A-SSE.2, A-APR.2, A-APR.3, F-IF.7c). Students continue to build upon the reasoning process of solving equations as they solve polynomial, rational, and radical equations, as well as linear and non-linear systems of equations (A-REI.1, A-REI.2, A-REI.6, A-REI.7). The module culminates with the fundamental theorem of algebra as the ultimate result in factoring. Connections to applications in prime numbers in encryption theory, Pythagorean triples, and modeling problems are pursued.

An additional theme of this module is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. Students use appropriate tools to analyze the key features of a graph or table of a polynomial function and relate those features back to the two quantities in the problem that the function is modeling (F-IF.7c).

**References:**
- EngageNY.org
- 2008 Louisiana State Comprehensive Curriculum
- Common Core State Standards
- As well as resources from the following states: Ohio, Tennessee, North Carolina, Connecticut, Arizona
### Enduring Understandings and Essential Questions for the Mathematical Practices:

<table>
<thead>
<tr>
<th>Mathematical Practice</th>
<th>Enduring Understanding</th>
<th>Essential Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1 Makes sense of problems and preserve in solving them</td>
<td>Solving a mathematical problem involves making sense of what is known and applying a thoughtful and logical process which sometimes requires perseverance, flexibility, and a bit of ingenuity</td>
<td>Mathematical Practice: How can <em>what is known</em> help determine <em>where to begin</em> or <em>what to do next</em> when solving a problem?</td>
</tr>
<tr>
<td>#2 Reason abstractly and quantitatively</td>
<td>The concrete and the abstract can complement each other in the development of mathematical understanding: representing a concrete situation with symbols can make the solution process more efficient, while reverting to a concrete context can help make sense of the abstract symbols</td>
<td>Mathematical Practice: How can both concrete and abstract reasoning help the solution process?</td>
</tr>
<tr>
<td>#3 Construct viable arguments and critique the reasoning of others</td>
<td>A well-crafted argument/critique requires a thoughtful and logical progression of mathematically sound statements and supporting evidence</td>
<td>Mathematical Practice: How can mathematical reasoning be supported?</td>
</tr>
<tr>
<td>#4 Model with mathematics</td>
<td>Many everyday problems can be solved by modeling the situation with mathematics</td>
<td>Mathematical Practice: How can a mathematical model help the solution process?</td>
</tr>
<tr>
<td>#5 Use appropriate tools strategically</td>
<td>Strategic choice and use of the tools can increase reliability and precision of results, enhance arguments, and deepen mathematical understanding</td>
<td>Mathematical Practice: What makes a tool the “best” tool for the job?</td>
</tr>
<tr>
<td>#6 Attend to precision</td>
<td>Attending to precise detail increases reliability of mathematical results and minimizes miscommunication of mathematical explanations.</td>
<td>Mathematical Practice: What makes the work clear and precise so that results are reliable and communicated effectively?</td>
</tr>
<tr>
<td>#7 Look for and make use of structure</td>
<td>Recognizing a structure or pattern can be the key to solving a problem or making sense of a mathematical idea.</td>
<td>Mathematical Practice: How can identifying a pattern or structure help the solution process?</td>
</tr>
<tr>
<td>#8 Look for and express regularity in repeated reasoning</td>
<td>Recognizing repetition or regularity in the course of solving a problem (or series of similar problems) can lead to results more quickly and efficiently.</td>
<td>Mathematical Practice: How can recognizing repetition or regularity help solve problems more efficiently?</td>
</tr>
</tbody>
</table>
Concept: Using Area Model to Represent Polynomial Multiplication

Estimated Time: 2 days

Common Core State Standard
Rewrite rational expressions
A-APR.6 Rewrite simple rational expressions in different forms; write \( \frac{a(x)}{b(x)} \) in the form \( q(x) + \frac{r(x)}{b(x)} \), where \( a(x) \), \( b(x) \), \( q(x) \), and \( r(x) \) are polynomials with the degree of \( r(x) \) less than the degree of \( b(x) \), using inspection, long division, or, for the more complicated examples, a computer algebra system.

Perform arithmetic operations on polynomials
A-APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Skills
Using the Area to Represent Polynomial Multiplication

Examples
Find the total area of the region: (Note: the inside expressions would not be given to students)
\[ A = 4x^3 + 10x^2 + 20x \]

Given:
\[ (x + 4)(2x + 2) \]

Draw an area diagram and find the total area.

Strategies, Resources, Etc:
- Use of Algebra tiles to demonstrate
- Activity that could be used to teach or a task: [http://www.doe.state.la.us/lde/uploads/1989.pdf](http://www.doe.state.la.us/lde/uploads/1989.pdf)
- Practice sheet: (also shown below) [http://www.bownet.org/ajumper/IntGeoPracticeMultPolysByAreaModel(08).pdf](http://www.bownet.org/ajumper/IntGeoPracticeMultPolysByAreaModel(08).pdf)
- Sample work and worksheet pasted below
- Puzzle Review activity pasted below
Worksheet: Multiplication of Polynomials by Area Model

Create an area model for each multiplication problem and expand each expression.

1. \((x + 5)(x - 2)\\)

2. \((x - 7)(x - 8)\\)

3. \((x + 1)(x - 6)\\)

4. \((x - 5)(x + 5)\\)

5. \((x - 7)(x + 7)\\)

6. \((x - 2)(x + 2)\\)

7. \((x - 4)(x + 4)\\)

8. \((x + 2)(x + 2)\\)

9. \((x + 3)(x + 3)\\)

10. \((3x^2 + 2)(2x)\\)

11. \((3x^2)(x - 2)\\)
Activity - Part 1
Have the students find the area of each smaller rectangle within the larger rectangle.

Example 1: \((x + 1)(x + 2)\)

\[
\begin{array}{c|c}
\text{X} & 2 \\
\hline
1 & \\
\end{array}
\quad \begin{array}{c|c}
\text{X} & 2 \\
\hline
x(x)=x^2 & x(2)=2x \\
1(x)=x & 1(2)=2
\end{array}
\]

Have the students add up all of the smaller areas to find the total area of the larger rectangle.

\[(x + 1)(x + 2) = x^2 + 2x + 1x + 2 = x^2 + 3x + 2\]

Example 2: \((x + 3)(x + 2)\)

\[
\begin{array}{c|c}
3 & 2 \\
\hline
x & \\
\end{array}
\quad \begin{array}{c|c}
3(x)=3x & 3(2)=6 \\
\hline
x(x)=x^2 & x(2)=2x
\end{array}
\]

Add up all of the smaller areas to find the total larger area.

\[(x + 3)(x + 2) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6\]
Example 3: \((x - 1)(x + 2)\)

\[
\begin{array}{c|c}
 x & 2 \\
\hline
 x & \\
-1 & \\
\end{array}
\quad=
\begin{array}{c|c}
 x & 2 \\
\hline
 x(x)=x^2 & x(2)=2x \\
 x(-1)=-1x & -1(2)=-2 \\
\end{array}
\]

Have the students add up all of the smaller areas to find the larger area.

\((x - 1)(x + 2) = x^2 + 2x - 1x - 2 = x^2 + 1x - 2\)

Classwork Answers:

A) \((x + 3)(x + 5) = x^2 + 8x + 15\)
B) \((x - 3)(x + 4) = x^2 + 1x - 12\)
C) \((x - 2)(x - 2) = x^2 - 4x + 4\)
D) \((x + 1)(2x - 1) = 2x^2 + 1x - 1\)
Activity- Part 1 Classwork
Work each problem with your group members.

A) \((x + 3)(x + 5)\)

\[
\begin{array}{ccc}
 x & 5 \\
 3 & & \\
 4 & & \\
\end{array}
\]

B) \((x - 3)(x + 4)\)

\[
\begin{array}{ccc}
 x & -3 \\
 4 & & \\
 3 & & \\
\end{array}
\]

C) \((x - 2)(x - 2)\)

\[
\begin{array}{ccc}
 x & -2 \\
 2 & & \\
 -2 & & \\
\end{array}
\]

D) \((x + 1)(2x - 1)\)

\[
\begin{array}{ccc}
 2x & -1 \\
 1 & & \\
 x & & \\
\end{array}
\]
### Multiplying Polynomials

Cut the squares apart.
Match equivalent expressions.
You should get a new 4 X 4 square.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x + 2)(x - 2))</td>
<td>((x + 3)^2)</td>
</tr>
<tr>
<td>((5x - 4)^2)</td>
<td>(x^2 + 6x + 9)</td>
</tr>
<tr>
<td>((x + 3)^2)</td>
<td>((x - 4)(x - 6))</td>
</tr>
<tr>
<td>((x - 2)(x - 9))</td>
<td>(4x^2 - 25)</td>
</tr>
<tr>
<td>((2x - 5)(2x + 5))</td>
<td>((x + 3)(x - 3))</td>
</tr>
<tr>
<td>((x + 2)(x + 5))</td>
<td>((k + 2)(x - 8))</td>
</tr>
<tr>
<td>((x + 3)(x + 1))</td>
<td>((7x - 5)(x - 2))</td>
</tr>
<tr>
<td>((2x - 5)(2x - 1))</td>
<td>((2x + 5)^2)</td>
</tr>
</tbody>
</table>

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Concept: Division of Polynomials by Monomials

Estimated Time: 1 day

Common Core State Standard

Rewrite rational expressions

A-APR.6 Rewrite simple rational expressions in different forms; write \( a(x)/b(x) \) in the form \( q(x) + r(x)/b(x) \), where \( a(x) \), \( b(x) \), \( q(x) \), and \( r(x) \) are polynomials with the degree of \( r(x) \) less than the degree of \( b(x) \), using inspection, long division, or, for the more complicated examples, a computer algebra system.

Perform arithmetic operations on polynomials

A-APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Objectives/Skills

- Division of Polynomial

Examples

Simplify \( \frac{2x + 4}{2} \)

One method:

\[
\frac{2x + 4}{2} = \frac{2x}{2} + \frac{4}{2} = \frac{2x}{2} + \frac{2}{2} = x + 2
\]

or this method:

\[
\frac{2x + 4}{2} = \frac{2(x + 2)}{2} = \frac{x(x + 2)}{x} = x + 2
\]

Simplify \( \frac{21x^3 - 35x^2}{7x} \)

Method 1:

\[
\frac{21x^3 - 35x^2}{7x} = \frac{21x^3}{7x} - \frac{35x^2}{7x} = 3x^2 - 5x
\]

or Method 2:

\[
\frac{21x^3 - 35x^2}{7x} = \frac{7x(3x^2 - 5x)}{7x} = 3x^2 - 5x
\]

Simplify:

\[
\frac{6x^2 - 3x - 3}{2x - 1}
\]

\[
\frac{3(2x^2 - x - 1)}{2x + 1}
\]

\[
\frac{3(x - 1)(2x + 1)}{2x + 1}
\]

\[
3(x - 1) = 3x - 3
\]

Instructional Strategies

Concept: Long Division of Polynomials

Estimated Time: 2 days

Common Core State Standard

Rewrite rational expressions

A-APR.6 Rewrite simple rational expressions in different forms; write \( \frac{a(x)}{b(x)} \) in the form \( q(x) + \frac{r(x)}{b(x)} \), where \( a(x) \), \( b(x) \), \( q(x) \), and \( r(x) \) are polynomials with the degree of \( r(x) \) less than the degree of \( b(x) \), using inspection, long division, or, for the more complicated examples, a computer algebra system.

Objectives/Skills

- Divide Polynomials using long division
- Explain what the “factors” of a polynomial

### Examples

<table>
<thead>
<tr>
<th>( \frac{x^2 + 9x + 14}{x + 7} )</th>
<th>( \frac{x^3 - 3x^2 + 6x - 4}{2x - 3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{x + 2}{x + 7} ) x^2 + 9x + 14</td>
<td>( \frac{2x^4 - 9x^3 + 21x^2 - 26x + 12}{2x^4 - 3x^3} )</td>
</tr>
<tr>
<td>( -\frac{x^2 + 7x}{x + 7} ) 2x + 14</td>
<td>( -6x^3 + 21x^2 - 26x + 12 )</td>
</tr>
<tr>
<td>( -\frac{2x + 14}{x + 7} ) -2x + 14</td>
<td>( -(-6x^3 + 9x^2) )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( 12x^2 - 26x + 12 )</td>
</tr>
<tr>
<td></td>
<td>( -(12x^2 - 18x) )</td>
</tr>
<tr>
<td></td>
<td>( -8x + 12 )</td>
</tr>
<tr>
<td></td>
<td>( -(-8x + 12) )</td>
</tr>
<tr>
<td></td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

### Strategies, Resources, Etc:

- Sample problems, video demo, and attached quiz: [http://www.mathsisfun.com/algebra/polynomials-division-long.html](http://www.mathsisfun.com/algebra/polynomials-division-long.html)
- Sample Notes and examples: [http://tutorial.math.lamar.edu/Classes/Alg/DividingPolynomials.aspx](http://tutorial.math.lamar.edu/Classes/Alg/DividingPolynomials.aspx)
- GENERATE a worksheet at various difficulty levels (Answers can also be generated): [http://www.softschools.com/math/algebra/polynomials/polynomial_long_division_worksheets/](http://www.softschools.com/math/algebra/polynomials/polynomial_long_division_worksheets/)
- Notes and examples: [http://www.sosmath.com/algebra/factor/fac01/fac01.html](http://www.sosmath.com/algebra/factor/fac01/fac01.html)
Divide each of the polynomials using long division.

1. \((4x^2 - 9) \div (2x + 3)\) 

2. \((x^2 - 4) \div (x + 4)\)

3. \((2x^2 + 5x - 3) \div (x + 3)\)

4. \((2x^2 + 5x - 3) \div (x - 3)\)

5. \((3x^2 - 13x - 10) \div (x - 5)\)

6. \((3x^2 - 13x - 10) \div (x + 5)\)

7. \((11x + 20x^2 + 12x^3 + 2) \div (3x + 2)\)

8. \((12x^3 + 2 + 11x + 20x^2) \div (2x + 1)\)

9. \(\frac{x^4 - 1}{x^2 - 1}\)

10. \(\frac{x^4 - 9}{x^2 + 3}\)
Concept: Synthetic Division
Estimated Days: 1 day

Common Core State Standard
Rewrite rational expressions

A-APR.6 Rewrite simple rational expressions in different forms; write \( a(x)/b(x) \) in the form \( q(x) + r(x)/b(x) \), where \( a(x) \), \( b(x) \), \( q(x) \), and \( r(x) \) are polynomials with the degree of \( r(x) \) less than the degree of \( b(x) \), using inspection, long division, or, for the more complicated examples, a computer algebra system.

Objectives/Skills
- Use synthetic division to show factors of polynomials

Examples
NOTE: At this point, your goal is NOT to have a remainder. Thus, choose only problems that do not have a remainder.

\[
\begin{array}{c|cccc}
\text{3} & 0 & -5 & 7 & -14 \\
\hline
\text{3} & 0 & 12 & -14 & 14 \\
\end{array}
\]

Solution (Quotient):
\( 3x^3 - 6x^2 + 7x - 7 \)

1. With synthetic division you always use “c” where the divisor is “\( x - c \)”. This is where we get “-2” – when put back into “x-c” form, we would have: \( x - (-2) \), where \( c = -2 \)
2. Use the coefficients of each of the terms
3. Always bring down the first term, multiply it by “c”
4. Bring your answer up and add it to the next column
5. Repeat multiplication and addition steps until finished.
6. Re-write polynomial in factored form

(Above example take from: http://www2.ccs.k12.va.us/teachers/McClelland/Alg3/Synthetic.pdf)

Strategies, Resources, Etc:
- Sample problems, answers and complete solution can be revealed (great for in-class practice): http://www.mesacc.edu/~scotz47781/mat120/notes/divide_poly/synthetic/synthetic_division_practice.html
- Sample problems, notes: http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut37_syndiv.htm
- In-class problems for practice, answers can be revealed: http://emathlab.com/Algebra/PolyFunctions/SyntheticDiv.php
- Notes and example demonstration: http://tutorial.math.lamar.edu/Classes/Alg/DividingPolynomials.aspx
- Unit 5, Activity 11: Synthetic Division (pasted below)
Activity 11: Synthetic Division (GLEs: Grade 9: 5; Grade 10: 2; Grade 11/12: 2, 5)

Materials List: paper, pencil

In this activity, students will use synthetic division to divide a polynomial by a first-degree binomial.

Math Log Bellringer:

Divide by hand to simplify the following quotients:

(1) $7 \div 5 = 1$ Remainder 2 or $7 \div 5 = \frac{2}{5}$

Solutions: (1) $191 \frac{5}{7}$ (2) $x^3 + 6x + 5 - \frac{4}{x - 2}$, (3) See Activity for discussion

Activity:

- Use Bellringer #1 to review elementary school terminology: $\frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}$.

  Rewrite this rule in Algebra II form: $\frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient} + \text{remainder}}{\text{divisor}}$ and relate to Bellringer problem 2.

- Review the definition of polynomial and the steps for long division, stressing descending powers and missing powers. Have students divide $\frac{2x^3 + 3x + 100}{x + 4}$.

  Solution: $x + 4 \left( \frac{2x^3 + 3x + 100}{x + 4} \right)$ with a remainder of $-40 = 2x^2 - 8x + 35 - \frac{40}{x + 4}$

- Introduce synthetic division illustrating that in the long division problems, the variable is not necessary, and if we had divided by the opposite of 4, we could have used addition instead of subtraction. Rework the problems using synthetic division.

  Solution: $-4 | 2 \ 0 \ 3 \ 100$

  $\begin{array}{r|rrr}
  & 0 & 3 & 100 \\
  \hline 
  -8 & & -52 & -140 \\
  \hline 
  2 & -8 & 35 & -40 \\
  \end{array}$

- Have students develop the steps for synthetic division:
  (1) Set up the coefficients in descending order of exponents.
  (2) If a term is missing in the dividend, write a zero in its place.
  (3) When dividing by the binomial $x - c$, use $c$ as the divisor ($c$ is the value of $x$ that makes the factor $x - c = 0$).
  (4) When dividing by the binomial $ax - c$, use $\frac{c}{a}$ as the divisor. ($\frac{c}{a}$ is the value of $x$ that makes the factor $ax - c = 0$.)

- Have students practice the use of synthetic division to simplify the following and write the answers in equation form as $\frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}$.

  (1) $\frac{2x^3 + 5x^2 - 7x - 12}{x + 3}$
  (2) $\frac{x^4 - 5x^2 - 10x - 12}{x + 2}$

  Solutions:

  $\begin{align*}
  (1) \quad \frac{2x^3 + 5x^2 - 7x - 12}{x + 3} &= (2x^2 - x - 4) + \frac{0}{x + 3} \\
  (2) \quad \frac{x^4 - 5x^2 - 10x - 12}{x + 2} &= x^3 - 2x^2 - x + 8 + \frac{4}{x + 2}
  \end{align*}$

- Use the math textbook for additional problems.
## Concept: The Factor Theorem

**Estimated Days:** 1 day

<table>
<thead>
<tr>
<th>Common Core State Standard</th>
<th>Estimated Days: 1 day</th>
</tr>
</thead>
<tbody>
<tr>
<td>Understand the relationship between zeros and factors of polynomials</td>
<td></td>
</tr>
</tbody>
</table>

**A-APR.2** Know and apply the Remainder Theorem: For a polynomial \( p(x) \) and a number \( a \), the remainder on division by \( x - a \) is \( p(a) \), so \( p(a) = 0 \) if and only if \( (x - a) \) is a factor of \( p(x) \).

**A-APR.3** Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

### Objectives/Skills

- Show that \( x - a \) and \( x - b \) is a factor of the polynomial using the Factor Theorem

### Examples

**NOTE:** Remainder Theorem is later in this Module. Some resources include both Remainder and Factor Theorem. Only Factor Theorem is covered at this time.

**Use the Factor Theorem to determine whether \( x - 1 \) is a factor of**

\[
f(x) = 2x^4 + 3x^2 - 5x + 7
\]

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>0</th>
<th>3</th>
<th>-5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

\( x - 1 \) is not a factor of \( f(x) \)

**Using the Factor Theorem, verify that \( x + 4 \) is a factor of**

\[
f(x) = 5x^4 + 16x^3 - 15x^2 + 8x + 16
\]

<table>
<thead>
<tr>
<th>-4</th>
<th>5</th>
<th>16</th>
<th>-15</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-20</td>
<td>16</td>
<td>-4</td>
<td>-16</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-4</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

\( x + 4 \) is a factor of \( 5x^4 + 16x^3 - 15x^2 + 8x + 16 \)

### Strategies, Resources, Etc:

- Bottom of this worksheet has Factor Theorem Quick note and sample problems: [http://www.math.wisc.edu/~algcoord/09spring/pdfs/noteshulls/noteshell_7_3.pdf](http://www.math.wisc.edu/~algcoord/09spring/pdfs/noteshulls/noteshell_7_3.pdf)
- Section 7-6: Rational Zero Theorem (p. 378)
Concept: Solve Quadratic Equations by Factoring

Estimated day: 3 days

Common Core State Standard

A.REI.4 – a. use the method of completing the square to transform any quadratic equation in x into an equation of the form \((x - p)^2 = q\) that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for \(x^2 = 49\)), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as \(a \pm bi\) for real numbers \(a\) and \(b\).

A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

- Factor a quadratic expression to reveal the zeros of the function it defines.

Objectives/Skills

- Factor a quadratic expression to find the zeros of the function it represents.
- Write a quadratic equation with given zeros.

Examples

1. \(x^2 = 6x\)
2. \(x^2 - 16x + 64 = 0\)
3. \(x^2 = 64\)
4. \(5x^2 - 5x - 60 = 0\)
5. \(4x^2 - 13x = 12\)

Instructional Strategies


Strategies, Resources, Etc:

**Comprehensive Curriculum:**

- Unit 2, Activity 6 (Pasted below)

**Textbook:**

- Section 6.3 Solving Quadratic Equation by Factoring

**Websites:**

- [http://teachers.henrico.k12.va.us/math/hcpsalgebra1/module8-6.html](http://teachers.henrico.k12.va.us/math/hcpsalgebra1/module8-6.html)
- [http://teachers.henrico.k12.va.us/math/hcpsalgebra2/6-2.htm](http://teachers.henrico.k12.va.us/math/hcpsalgebra2/6-2.htm)
- [http://www.purplemath.com/modules/solvquad.htm](http://www.purplemath.com/modules/solvquad.htm)
- [http://www.webmath.com/quadtri.html](http://www.webmath.com/quadtri.html)

**External Resources:**

  (includes videos in Spanish)

**Remediation:**

- [http://www.kutasoftware.com/freeia2.html](http://www.kutasoftware.com/freeia2.html)
Activity 6: Solving Equations by Factoring

Materials List: paper, pencil, graphing calculator

In this activity, the students will develop the Zero–Product Property and use it and their factoring skills to solve polynomial equations.

Math Log Bellringer:
Solve for $x$:
1. $2x = 16$
2. $2x^2 = 16x$
3. Explain the property used to get the answer to #2.

Solutions:
1. $x = 8$
2. $x = 8$ and $x = 0$
3. There will be several explanations. See the activity below to guide students to the correct explanation.

Activity:

- Determine how many students got both answers in Bellringer problem #2 and use this to start a discussion about division by a variable—do not divide both sides of an equation by a variable because the variable may be zero.

  Define division as $\frac{a}{b} = c$ if and only if $bc = a$ and have students explain why division by zero is “undefined.”

- Have a student who worked problem #2 correctly, write the problem on the board showing his/her work. (He/she should have isolated zero and factored.) Have the students develop the Zero–Product Property of Equality. Make sure students substitute to check their answers. Review the use of and or in determining the solution sets in compound sentences. Compare the solution for problem #2 with the solution to the problem $x(x + 2) = 8$ solved incorrectly as $\{8, 6\}$. Have students substitute solutions to check answers and discuss why there is no “Eight Product Property of Equality” or any other number except zero. Use guided practice to allow students to solve several more quadratic polynomial equations using factoring.

- Have the students solve the following and discuss double and triple roots and multiplicity. Multiplicity occurs when the same number is a solution more than once.

  1. $(x - 4)(x - 3)(x + 2) = 0$
  2. $y^3 - 3y^2 = 10y$
  3. $x^2 + 6x = -9$
  4. $(x^2 + 4x + 4)(x + 2) = 0$

     Solutions:

     1. $\{-2, 3, 4\}$
     2. $\{0, 5, -2\}$
     3. There is one solution with multiplicity of 2; therefore, the solution is called a double root.
     4. $\{-2\}$, There is one solution with multiplicity of 3; therefore, the solution is called a triple root.

- Have students develop the steps for solving an equation by factoring:

  Step 1: Write in Standard Form (Isolate zero)
  Step 2: Factor
  Step 3: Use the Zero–Product Property of equality
  Step 4: Find the solutions
  Step 5: Check

- Application:

  Divide the students in groups to set up and solve these application problems:

  1. The perimeter of a rectangle is 50 in. and the area is 144 in$^2$. Find the dimensions of the rectangle.
(2) A concrete walk of uniform width surrounds a rectangular swimming pool. Let \( x \) represent this width. If the pool is 6 ft. by 10 ft. and the total area of the pool and walk is 96 ft\(^2\), find the width of the walk.

(3) The longer leg of a right triangle has a length 1 in. less than twice the shorter leg. The hypotenuse has a length 1 in. greater than the shorter leg. Find the length of the three sides of the triangle.

*Solutions:*

(1) 16 in. by 9 in., (2) 1 foot, (3) 2.5 in., 2 in., and 1.5 in.
### Concept: Graphing Quadratic Functions

**Time Frame:** 3 days

#### Common Core State Standard

Analyze functions using different representations

F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

Graph linear and quadratic functions and show intercepts, maxima, and minima.

#### Objectives/Skills

- Define and graph a quadratic function
- Identify the parts of a quadratic function
  - Identify and write the equation of the axis of symmetry.
  - Identify the y-intercept.
  - Identify the vertex.
  - Find the maximum and minimum values.

#### Examples

1. \( f(x) = 2x^2 \)
2. \( f(x) = -x^2 - 2x + 3 \)
3. \( f(x) = x^2 + 3x - 4 \)

#### Instructional Strategies


#### Resources

**Comprehensive Curriculum:**

- Unit 5, Activity 1: Zeros *(pasted below)*
- Unit 5, Activity 2: The Vertex and Axis of Symmetry *(pasted below)*

**Textbook:**

- Sec 6.1: Graphing Quadratic Equations (p. 286)

**Websites:**

- [www.kutasoftware.com](http://www.kutasoftware.com) – use for pre-made graphs and worksheets
- [http://www.purplemath.com/modules/grphquad.htm](http://www.purplemath.com/modules/grphquad.htm) -- tutorial and notes
- [http://www.youtube.com/watch?v=mDwN1SqnMRU](http://www.youtube.com/watch?v=mDwN1SqnMRU) -- tutorial with examples

**Remediation:**

- Algebra I textbook, Chapter 9: Factoring (breaks down into individual factoring techniques)
- Algebra I textbook, Section 10.1

**Enrichment:**

- [http://www.analyzemath.com/Graphing/GraphingQuadraticFunction.html](http://www.analyzemath.com/Graphing/GraphingQuadraticFunction.html) -- extends graphing with domain and range in interval notation

**External Resources:**

- [http://www.valleymv.k12.oh.us/vvhs/dept/math/quadhelp.html](http://www.valleymv.k12.oh.us/vvhs/dept/math/quadhelp.html) -- tutorial Quadratic Functions attached at the end
Activity 1: Why Are Zeroes of a Quadratic Function Important?

Materials List: paper, pencil, graphing calculator, Math Log Bellringer BLM, Zeroes of a Quadratic Function BLM

In this activity, the students will plot data that creates a quadratic function and will determine the relevance of the zeroes and the maximum and minimum of values of the graph. They will also examine the sign and magnitude of the leading coefficient in order to make an educated guess about the regression equation for some data. By looking at real-world data first, the symbolic manipulations necessary to solve quadratic equations have significance.

Math Log Bellringer:

One side (s) of a rectangle is four inches less than the other side. Draw a rectangle with these sides and find an equation for the area A(s) of the rectangle.

\[ A(s) = s(s - 4) = s^2 - 4s \]

Activity:

- Overview of the Math Log Bellringers:
  - As in previous units, each in-class activity in Unit 5 is started with an activity called a Math Log Bellringer that either reviews past concepts to check for understanding (reflective thinking about what was learned in previous classes or previous courses) or sets the stage for an upcoming concept (predictive thinking for that day’s lesson).
  - A math log is a form of a learning log (view literacy strategy descriptions) that students keep in order to record ideas, questions, reactions, and new understandings. Documenting ideas in a log about content being studied forces students to “put into words” what they know or do not know. This process offers a reflection of understanding that can lead to further study and alternative learning paths. It combines writing and reading with content learning. The Math Log Bellringers will include mathematics done symbolically, graphically, and verbally.
  - Since Bellringers are relatively short, blackline masters have not been created for each of them. Write them on the board before students enter class, paste them into an enlarged Word™ document or PowerPoint™ slide, and project using a TV or digital projector, or print and display using a document or overhead projector. A sample enlarged Math Log Bellringer Word™ document has been included in the blackline masters. This sample is the Math Log Bellringer for this activity.
  - Have the students write the Math Log Bellringers in their notebooks, preceding the upcoming lesson during beginning-of-class record keeping, and then circulate to give individual attention to students who are weak in that area.

- Use the Bellringer to relate second-degree polynomials to the name “quadratic” equations (area of a quadrilateral). Discuss the fact that this is a function and have students identify this shape as a parabola.

- Zeroes of a Quadratic Function BLM:
  - Distribute the Zeroes of a Quadratic Function BLM. This is a teacher/student interactive worksheet. Stop after each section to clarify, summarize, and stress important concepts.
Zeroes: Review the definition of zeroes from Unit 2 as the x-value for which the y-value is zero, thus indicating an x-intercept. In addition to the answers to the questions, review with the students how to locate zeroes and minimum values of a function on the calculator. (TI–83 and 84 calculator: Graph Calc (2nd Trace) 2: zero or 3: minimum)

Local and Global Characteristics of a Parabola: In Activity 2, the students will develop the formulas for finding the vertex and the equation of the axis of symmetry. In this activity, students are simply defining, identifying, and reviewing domain and range.

Reviewing 2nd Degree Polynomial Graphs: Review the concepts of end-behavior, zeroes and leading coefficients.

Application: Allow students to work this problem in groups to come to a consensus. Have the students put their equations on the board or enter them into the overhead calculator. Discuss their differences, the relevancy of the zeroes and vertex, and the various methods used to solve the problem. Discuss how to set up the equation from the truck problem to solve it analytically. Have the students expand, isolate zero, and find integral coefficients to lead to a quadratic equation in the form $y = ax^2 + bx + c$. Graph this equation and find the zeroes on the calculator. This leads to the discussion of the reason for solving for zeroes of quadratic equations.
One side, $s$, of a rectangle is four inches less than the other side.

Draw a rectangle with these sides and find an equation for the area $A(s)$ of the rectangle.
Graph the function from the Bellringer $y = x^2 - 4x$ on your calculators. This graph is called a parabola. Sketch the graph making sure to accurately find the $x$ and $y$ intercepts and the minimum value of the function.

(1) In the context of the Bellringer, what do the $x$–values represent? ________________  
the $y$–values? ________________

(2) From the graph, list the zeroes of the equation. ________________

(3) What is the real-world meaning of the zeroes for the Bellringer?

(4) Solve for the zeroes analytically showing your work. What property of equations did you use to find the zeroes?

**Local and Global Characteristics of a Parabola**

(1) In your own words, define axis of symmetry: ________________

(2) Write the equation of the axis of symmetry in the graph above. ________________

(3) In your own words, define vertex: ________________

(4) What are the coordinates of the vertex of this parabola? ________________

(5) What is the domain of the graph above? ________________ range? ________________

(6) What domain has meaning for the Bellringer and why? ________________

(7) What range has meaning for the Bellringer and why? ________________
Reviewing 2nd Degree Polynomial Graphs

Graph the following equations and answer the questions in your notebook.

1. \( y = x^2 \) and \( y = -x^2 \). How does the sign of the leading coefficient affect the graph of the parabola?

2. \( y = x^2, y = 4x^2, y = 0.5x^2 \). How does the magnitude of the leading coefficient affect the zeroes and the shape of the parabola as compared to \( y = x^2 \)?

3. \( y = (x - 3)(x + 4), y = (x - 1)(x + 6) \). Make conjectures about the zeroes.

4. \( y = 2(x - 5)(x + 4), y = -2(x - 5)(x + 4) \). Make conjectures about the zeroes and end-behavior.

Application

A tunnel in the shape of a parabola over a two-lane highway has the following features. It is 30 feet wide at the base and 23 feet high in the center.

1. Make a sketch of the tunnel on a coordinate plane with the ground as the x-axis and the left side of the base of the tunnel at (2, 0). Find two more ordered pairs and graph as a scatter plot in your calculator.

2. Enter the quadratic equation \( y = a(x - b)(x - c) \) in your calculator substituting your x-intercepts from your sketch into b and c. Experiment with various numbers for \( a \) to find the parabola that best fits this data. Write your equation.

3. An 8-foot wide 12-foot high truck wants to go through the tunnel. Determine whether the truck will fit and the allowable location of the truck. Explain your answer.
Graph the function from the Bellringer $y = x^2 - 4x$ on your calculators. This graph is called a parabola. Sketch the graph making sure to accurately find the $x$ and $y$ intercepts and the minimum value of the function.

(1) In the context of the Bellringer, what do the $x$–values represent? **the length of the sides** the $y$–values? **the area**

(2) From the graph, list the zeroes of the equation. 0 and 4

(3) What is the real-world meaning of the zeroes for the Bellringer? **The length of the side for which the area is zero.**

(4) Solve for the zeroes analytically showing your work. What property of equations did you use to find the zeroes?

\[ 0 = x^2 - 4x \Rightarrow 0 = x(x - 4) \Rightarrow x = 0 \text{ or } x - 4 = 0 \text{ by the Zero Property of Equations} \Rightarrow \{0, 4\} \]

**Local and Global Characteristics of a Parabola**

(1) In your own words, define axis of symmetry: **a line about which pairs of points on the parabola are equidistant**

(2) Write the equation of the axis of symmetry in the graph above. \( x = 2 \)

(3) In your own words, define vertex: **The point where the parabola intersects the axis of symmetry**

(4) What are the coordinates of the vertex of this parabola? \((2, -4)\)

(5) What is the domain of the graph above? **all real numbers** range? \( y \geq -4 \)

(6) What domain has meaning for the Bellringer and why? \( x > 4 \text{ because these sides create positive area.} \)

(7) What range has meaning for the Bellringer and why? \( y > 0 \text{ because you want an area} > 0 \)
Module 1: Polynomial, Rational, and Radical Relationships

Reviewing 2nd Degree Polynomial Graphs

Graph the following equations and answer the questions in your notebook.

(1) \( y = x^2 \) and \( y = -x^2 \). How does the sign of the leading coefficient affect the graph of the parabola?
   
   Even exponent polynomial has similar end behavior. Positive leading coefficient starts up and ends up, negative leading coefficient starts down and ends down.

(2) \( y = x^2, y = 4x^2, y = 0.5x^2 \). How does the magnitude of the leading coefficient affect the zeroes and the shape of the parabola as compared to \( y = x^2 \)?
   
   It does not affect the zeroes. If constant is > 1, the graph is steeper than \( y = x^2 \), and if the coefficient is less than 1, the graph is wider than \( y = x^2 \).

(3) \( y = (x - 3)(x + 4), y = (x - 1)(x + 6) \). Make conjectures about the zeroes. When the function is factored, the zeroes of the parabola are at the solutions to the factors set = 0.

(4) \( y = 2(x - 5)(x + 4), y = -2(x - 5)(x + 4) \). Make conjectures about the zeroes and end-behavior. Same zeroes opposite end-behaviors.

Application

A tunnel in the shape of a parabola over a two-lane highway has the following features. It is 30 feet wide at the base and 23 feet high in the center.

(1) Make a sketch of the tunnel on a coordinate plane with the ground as the x-axis and the left side of the base of the tunnel at \((2, 0)\). Find two more ordered pairs and graph as a scatter plot in your calculator. \((32, 0) \) and \((17, 23)\)

(2) Enter the quadratic equation \( y = a(x - b)(x - c) \) in your calculator substituting your \( x \)-intercepts from your sketch into \( b \) and \( c \). Experiment with various numbers for “a” to find the parabola that best fits this data. Write your equation.

\[ y = -0.1(x - 2)(x - 32) \]

(3) An 8-foot wide 12-foot high truck wants to go through the tunnel. Determine whether the truck will fit and the allowable location of the truck. Explain your answer.

The truck must travel 4.75 feet from the base of the tunnel. It is 8 feet wide and the center of the tunnel is 15 feet from the base so the truck can stay in its lane.
Activity 2: The Vertex and Axis of Symmetry

Materials List: paper, pencil, graphing calculator

In this activity, the student will graph a variety of parabolas, discovering the changes that shift the graph vertically, horizontally, and obliquely, and will determine the value of the vertex and axis of symmetry.

Math Log Bellringer:

1. Graph \( y_1 = x^2 \), \( y_2 = x^2 + 4 \), and \( y_3 = x^2 - 9 \) on your calculator, find the zeroes and vertices, and write a rule for the type of shift \( f(x) + k \).

2. Graph \( y_1 = (x - 4)^2 \), \( y_2 = (x + 2)^2 \) on your calculator, find the zeroes and vertices, and write a rule for the type of shift \( f(x + k) \).

3. Graph \( y_1 = x^2 - 6x \) and \( y_2 = 2x^2 - 12x \) on your calculator. Find the zeroes and vertices on the calculator. Find the equations of the axes of symmetry. What is the relationship between the vertex and the zeroes? What is the relationship between the vertex and the coefficients of the equation?

Solutions:

1. Zeroes: \( y_1 \): \{0\}, \( y_2 \): none, \( y_3 \): \{±3\}. Shift up if \( k > 0 \) and down if \( k < 0 \).

2. Zeroes: \( y_1 \): \{4\}, \( y_2 \): \{−2\}. Vertices: \( y_1 \): \( (4, 0) \), \( y_2 \): \( (−2, 0) \). Shift right if \( k < 0 \), shift left if \( k > 0 \).

3. Zeroes: \( y_1 \): \{0, 6\}, \( y_2 \): \{0, 6\}. Vertices: \( y_1 \): \( (3, −9) \), \( y_2 \): \( (3, −18) \), axes of symmetry \( x = 3 \). The \( x \)-value of the vertex is the midpoint between the \( x \)-values of the zeroes. A leading coefficient changes the \( y \)-value of the vertex.

Activity:

- Use the Bellringer to begin the development of the formula for finding the vertex of a quadratic function in the form \( f(x) = ax^2 + bx \):
  - Have the students set \( ax^2 + bx \) equal to 0 to find the zeroes, 0 and \( \frac{-b}{a} \).
  - Have the students find the midpoint between the zeroes at \( \frac{-b}{2a} \) to find the \( x \)-value or abscissa of the vertex and the axis of symmetry.
  - Have the students substitute the abscissa into the equation \( f(x) = ax^2 + bx \) to find the ordinate of the vertex \( f \left( \frac{-b}{2a} \right) \).

- Assign problems from the textbook for students to apply the formula for the vertex \( \left( \frac{-b}{2a}, f \left( \frac{-b}{2a} \right) \right) \) to practice graphing functions in the form \( f(x) = ax^2 + bx + c \).

- Have the students develop a set of steps to graph a factorable quadratic function in the form \( f(x) = ax^2 + bx + c \) without a calculator:
1. Find the zeroes by factoring the equation and applying the Zero Property of Equations.

2. Find the vertex by letting \( x = \frac{-b}{2a} \) and \( y = f \left( \frac{-b}{2a} \right) \).

3. Graph and make sure that the graph is consistent with the end–behavior property that says, if \( a > 0 \) the graph opens up and if \( a < 0 \) it opens down.

**Application:**
The revenue, \( R \), generated by selling games with a particular price is given by \( R(p) = -15p^2 + 300p + 1200 \). Graph the revenue function without a calculator and find the price that will yield the maximum revenue. What is the maximum revenue? Explain in real world terms why this graph is parabolic.

\textit{Solution: } price = $10, maximum revenue = $2700. A larger price will generate more revenue until the price is so high that no one will buy the games and the revenue declines.
## Concept: Analyzing Graphs of Quadratic Functions

### Time Frame: 2 days

### Common Core State Standard(s)

#### Analyze functions using different representations

F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★

- a. Graph linear and quadratic functions and show intercepts, maxima, and minima.
- b. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

### Objectives/Skills:

- Determine how varying the coefficients in $y = ax^2 + bx + c$ affects the graph

### Examples:

1. Changing $a$:
   - a) $y = x^2$
   - b) $y = 2x^2$
   - c) $y = \frac{1}{2}x^2$

2. Changing $h$:
   - a) $y = x^2$
   - b) $y = (x - 2)^2$
   - c) $y = (x + 2)^2$

3. Changing $k$:
   - a) $y = x^2$
   - b) $y = x^2 + 3$
   - c) $y = x^2 - 3$

### Instructional Strategies:

- Use the calculator to show the difference when changing various variables in the equation. It helps if the students can discover the rules on their own.
- Graphing Calculator Investigation p.320
- Graphing Families of Quadratic Functions:
  [http://education.ti.com/calculators/downloads/US/Activities/Detail?id=7529&ref=%2fcalculators%2fdownloads%2fUS%2fActivities%2fSearch%2fSubject%3fs%3d5022%26sa%3d1010%26t%3d1179%26d%3d9](http://education.ti.com/calculators/downloads/US/Activities/Detail?id=7529&ref=%2fcalculators%2fdownloads%2fUS%2fActivities%2fSearch%2fSubject%3fs%3d5022%26sa%3d1010%26t%3d1179%26d%3d9)

### Resources

#### Comprehensive Curriculum:

- Unit 5, Activity 7: How varying the Coefficients in $y = ax^2 + bx + c$ affects the Graphs (pasted below)

#### Textbook:

- Section 6.6 Analyzing Graphs of Quadratic Functions (p.322)

#### Websites:

- [http://www.purplemath.com/modules/grphquad2.htm](http://www.purplemath.com/modules/grphquad2.htm)
- [http://www.youtube.com/watch?v=NCQgQCuyIYo](http://www.youtube.com/watch?v=NCQgQCuyIYo) -- further exploration with TI-Nspire

#### External Resources:

[http://www.youtube.com/watch?v=ISBhPIMiO3s](http://www.youtube.com/watch?v=ISBhPIMiO3s) -- video

### Activity 7: How Varying the Coefficients in $y = ax^2 + bx + c$ Affects the Graphs

In this activity, students will discover how changes in the equation for the quadratic function can affect the graph in order to create a best-fit parabola.

Math Log Bellringer:

Graph $y_1 = -4x + 6$ and $y_2 = x(-4x + 6)$ without a calculator, discuss similarities, then describe the method you used to graph the equations.

Solution:

Students should say that the graphs both have the same zero at $x = 3/2$. Answers to the discussion may vary. They could have graphed $y_1$ by finding the $y$-intercept and using the slope to graph, or they could have plotted points. Students could have found the zeroes in $y_2$ at $x = 0$ and $3/2$ by using the Zero Property of Equations or the quadratic formula, and they could have found the vertex by finding the midpoint between the zeroes or by using

$$
\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).
$$

Activity:

- Use the Bellringer to check for understanding of the relationship between $y = mx + b$ and $y = x(mx+b)$ before going on to other changes.
- Distribute the Graphing Parabolas Anticipation Guide BLM.
  - An anticipation guide is a modified form of opinionnaire (view literacy strategy descriptions) which promotes deep and meaningful understandings of content area topics by activating and building relevant prior knowledge, and by building interest in and motivation to learn more about particular topics. Anticipation guides are developed by generating statements about a topic that force students to take positions and defend them. The emphasis is on students’ points of view and not the “correctness” of their opinions.
  - In the Graphing Parabolas Anticipation Guide BLM, the students will use their prior knowledge of translating graphs to predict how changes in $a$, $b$, and $c$ in the equation $y = ax^2 + bx + c$ will affect the graph.
  - This should take approximately five minutes after which the students will discover exactly what happens to the graph using The Changing Parabola Discovery Worksheet BLM. There is no Graphing Parabolas Anticipation Guide with Answers BLM because the answers may vary based on the students’ opinions.
- The Changing Parabola Discovery Worksheet:
  - On this worksheet the students will use their graphing calculators to graph the parabola $y = ax^2 + bx + c$ with various changes in the constants to determine how these changes affect the graph, and they will compare their answers to the predictions in the anticipation guide.
  - Teach the following graphing technique before distributing the worksheet. Instead of graphing every equation individually, students can easily change the constants in one of three ways on
the TI83 and TI84 graphing calculator. Practice with the following example: \( y = x^2 + a \) for \( a = \{-2, 0, 4\} \)

1. Type three related equations: \( y_1 = x^2, y_2 = y_1 - 2, y_3 = y_1 + 4. \) (\( y_1 \) is found under \( \text{VARS, Y-VARS, 1:Function..., 1:Y}_1 \))

2. Use a list: \( y_1 = x^2 + \{-2, 0, 4\} \) (brackets are found above the parentheses.)

3. Use the Transformation APPS:
   - Turn on the application by pressing \( \text{APPS Transfm ENTER ENTER} \)
   - Enter the equation \( y_1 = x^2 + A \) (Use the letters A, B, C, or D for constant that will be changed.)
   - Set the window by pressing \( \text{WINDOW cursor to SETTINGs, set where A will start, in this example A = -2, and adjust the step for A to Step = 1.} \)
   - Graph and use the \( \mathbf{\Delta} \) cursor to change the values of A.

   - When finished, uninstall the transformation APP by pressing \( \text{APPS Transfm, 1:Uninstall} \)

   - For more information see the TI 83/TI84 Transformation App Guidebook at http://education.ti.com/downloads/guidebooks/eng/transgraph-eng.

- Distribute The Changing Parabola Discovery Worksheet BLM and arrange the students in pairs to complete it. Circulate to make sure they are graphing correctly.
- The answers to “why the patterns occur” will vary. When the students finish the worksheet, list the answers from the students on the board reviewing all the information they have learned in previous units, such as finding the vertex from \( \left( -\frac{b}{2a}, f\left( -\frac{b}{2a}\right) \right) \) and the axis of symmetry from \( x = -\frac{b}{2a} \), as well as, using the discriminant \( b^2 - 4ac \) to determine when there are real roots.

- Critical Thinking Writing Assessment: (See Activity-Specific Assessments at end of unit.)
Give your opinion of what will happen to the graphs in the following situations based upon your prior knowledge of translations and transformations of graphs.

(1) Predict what will happen to the graphs of form \( y = x^2 + 5x + c \) for the following values of \( c \): \{8, 4, 0, –4, –8\}.

(2) Predict what will happen to the graphs of form \( y = x^2 + bx + 4 \) for the following values of \( b \): \{6, 3, 0, –3, –6\}.

(3) Predict what will happen to the graphs of form \( y = ax^2 + 5x + 4 \) for the following values of \( a \): \{-2, –1, –\( \frac{1}{2} \), 0, \( \frac{1}{2} \), 1, 2\}.
Module 1: Polynomial, Rational, and Radical Relationships

Name ___________________________ Date ______________________

(1) Graph \( y = x^2 + 5x + 4 \) which is in the form \( y = ax^2 + bx + c \) (without a calculator). Determine the following global characteristics:

- Vertex: ___________ x–intercept: _______, y–intercept: ______
- Domain: ______________ Range: __________________________
- End–behavior: __________________________

(2) Graph \( y = x^2 + 5x + c \) on your calculator for the following values of \( c \): \( \{8, 4, 0, -4, -8\} \) and sketch. (WINDOW: \( x: [-10, 10] \), \( y: [-15, 15] \))

- What special case occurs at \( c = 0 \)? _______________________
- Check your predictions on your anticipation guide. Were you correct? Explain why the patterns occur.

(3) Graph \( y = x^2 + bx + 4 \) on your calculator for the following values of \( b \): \( \{6, 3, 0, -3, -6\} \) and sketch.

- What special case occurs at \( b = 0 \)? _______________________
- Check your predictions on your anticipation guide. Were you correct? Explain why the patterns occur.
(4) Graph \( y = ax^2 + 5x + 4 \) on your calculator for the following values of \( a: \{2, 1, 0.5, 0, -0.5, -1, -2\} \) and sketch.

- What special case occurs at \( a = 0? \)
- Check your predictions on your anticipation guide. Were you correct? Explain why the patterns occur.
(1) Graph \( y = x^2 + 5x + 4 \) which is in the form \( y = ax^2 + bx + c \) (without a calculator). Determine the following global characteristics:

**Vertex:** \( \left( -\frac{5}{2}, -\frac{9}{4} \right) \)  
**x–intercept:** \( \{-4, -1\} \)  
**y–intercept:** \( \{4\} \)

**Domain:** All Reals  
**Range:** \( y \geq -\frac{9}{4} \)

**End–behavior:** \( \text{as } x \to \pm \infty, y \to \infty \)

(2) Graph \( y = x^2 + 5x + c \) on your calculator for the following values of \( c \): \{8, 4, 0, -4, -8\} and sketch. (WINDOW: \( x: [-10, 10], y: [-15, 15] \))

- What special case occurs at \( c = 0 \)? The parabola passes through the origin.
- Check your predictions on your anticipation guide. Were you correct? Explain why the patterns occur. There are vertical shifts because you are just adding or subtracting a constant to the graph of \( y = x^2 + 5x \), so the \( y \) changes.

(3) Graph \( y = x^2 + bx + 4 \) on your calculator for the following values of \( b \): \{6, 3, 0, -3, -6\} and sketch.

- What special case occurs at \( b = 0 \)? The \( y\text{-axis is the axis of symmetry and the vertex is at } (0, 4) \)
- Check your predictions on your anticipation guide. Were you correct? Explain why the patterns occur. There are oblique shifts with the \( y\text{-intercept remaining the same, but the vertex is becoming } \text{more negative because the vertex is affected by } b \text{ found using } \left( -\frac{b}{2a}, f\left(-\frac{b}{2a}\right) \right) \text{ and } a \text{ is } 1. \text{ The axis of symmetry is } x = -\frac{b}{2a}, \text{ so when } b > 0, \text{ it moves left, and when } b < 0, \text{ the axis of symmetry moves right. Since real zeroes are determined by the discriminant } b^2 - 4ac \text{ which in this case is } b^2 - 16, \text{ when } |b| \geq 4, \text{ there will be real zeroes.}

(4) Graph \( y = ax^2 + 5x + 4 \) on your calculator for the following values of \( a \): \{2, 1, 0.5, 0, -0.5, -1, -2\} and sketch.
What special case occurs at \( a = 0 \)? The graph is the line \( y = 5x + 4 \).

Check your predictions on your anticipation guide. Were you correct? Explain why the patterns occur.
The \( y \)-intercept remains the same. When \( |a| > 1 \), the parabola is skinny and when \( |a| < 1 \) the parabola is wide. When \( a \) is positive, the parabola opens up; and when \( a \) is negative, the parabola opens down. The axis of symmetry is affected by \( a \), so as \( |a| \) gets bigger, the axis of symmetry approaches \( x = 0 \). Since real zeroes are determined by the discriminant, which in this case is \( 25 - 14a \), when \( |a| \geq \frac{25}{14} \) there will be real zeroes.
<table>
<thead>
<tr>
<th>Concept: Graphing Factored Cubics</th>
<th>Estimated Days: 2-3 days</th>
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<tbody>
<tr>
<td>Common Core State Standard</td>
<td></td>
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<tr>
<td>Understand the relationship between zeros and factors of polynomials</td>
<td></td>
</tr>
<tr>
<td>A-APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.</td>
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<tr>
<td>Objectives/Skills</td>
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<tr>
<td>• Sketch rough graphs of cubics through zeros, and y-intercept; noting any multiple roots</td>
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<tr>
<td>Examples</td>
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<tr>
<td>Do Mapshell lesson: <a href="http://www.map.mathshell.org/materials/download.php?fileid=1271">http://www.map.mathshell.org/materials/download.php?fileid=1271</a></td>
<td></td>
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<tr>
<td>Everything needed for this lesson is inclusive.</td>
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<td>Strategies, Resources, Etc:</td>
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<tr>
<td>• Use above mapshell lesson</td>
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<tr>
<td>• Use volume of cubes and rectangular prisms to demonstrate cubics</td>
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Quadratic Functions

Answer the following questions. Remember to be exact. (10 points each)

1. \( y = -2x^2 + 6x + 3 \)
   a. what is the min/max
   b. vertex
   c. axis of symmetry
   d. y –int.
   e. real zeros

2. \( y = 4x^2 - 8x + 2 \)
   a. what is the min/max
   b. vertex
   c. axis of symmetry
   d. y –int.
   e. real zeros

3. \( y = 4x^2 - 2x + 1 \)
   a. what is the min/max
   b. vertex
   c. axis of symmetry
   d. y –int.
   e. real zeros

Match each graph to one of the following functions. Place the letter next to the equation. (5 pts)

4. \( y = x^2 + 2 \) __________
5. \( y = -x^2 - 2 \) __________
6. \( y = x^2 - x + 2 \) __________
7. \( y = x^2 + x + 2 \) __________
8. \( y = x^2 - 2x + 2 \) __________
9. \( y = x^2 + x \) __________
10. \( y = x^2 - x \) __________
11. \( y = x^2 + 2x + 2 \) __________

A. ________________________________
   B. ________________________________
   C. ________________________________
   D. ________________________________

E. ________________________________
   F. ________________________________
   G. ________________________________
   H. ________________________________
Determine the quadratic equation whose graph is given. (5 pts. each)

12. 

13. 

Fill in the blanks. (3 points each)

14. The graph of a quadratic function is called a _______________________.

15. The vertical line passing through the vertex of a parabola dividing the parabola in half is called a(n) _______________________.

16. The point where the parabola crosses the y-axis is called the ____________________.

17. The highest y-value of a parabola opening down is called a(n) _____________________.

18. The lowest y-value of a parabola opening up is called a(n) _____________________.

World Problems (2 points each letter)

19. The price $p$ and $x$, the quantity of a pair of jeans sold, obey the demand equation

$$p = -\frac{1}{10}x + 100; 0 \leq x \leq 1000$$

a. Express the revenue $R$ as a function of $x$.

b. What is the revenue if 450 units are sold?

c. What quantity $x$ maximizes revenue?

d. What is the maximum revenue?

e. What price should the company charge to maximize revenue?

20. Kim is launched from a cliff 200 feet above the water at an inclination of 45° to the horizontal, with a muzzle velocity of 380 feet per second. The height $h$ of Kim above the water is given by
h(x) = \(-\frac{32x^2}{(180)^2}\) + x + 200, where x is the horizontal distance of Kim from the base of the cliff.

a) How far from the base of the cliff is the height of Kim a maximum?
b) Find the maximum height of Kim.
c) How far from the base of the cliff will Kim strike the water?
d) When the height of Kim is 100 feet above the water, how far is from the cliff?
e) When Kim is 1000 feet from the base of the cliff, what is the height of Kim?
f) Sketch the graph of Kim being launched into the air from the cliff.

21. A developer wants to enclose a rectangular grassy lot for parking and has 320 feet of fencing.
a) Express the area A of the rectangle as a function of the width x of the rectangle.
b) For what value of x is the largest area?
c) What is the maximum area?
d) What are the dimensions of the lot with maximum area?

Solve using the square root method.

1) \(x^2 = 25\)

2) \(x^2 = 36\)

3) \((x+1)^2 = 4\)

4) \((x+2)^2 = 1\)

5) \((2x+3)^2 = 9\)

6) \((3x-2)^2 = 4\)

What number should be added to complete the square?

7) \(x^2 + 8x\)
8) \( x^2 - 4x \)

9) \( x^2 + \frac{1}{2}x \)

10) \( x^2 + \frac{1}{3}x \)

11) \( x^2 - \frac{2}{3}x \)

12) \( x^2 - \frac{2}{5}x \)

Solve each equation by completing the square.

13) \( x^2 + 4x = 21 \)

14) \( x^2 - 6x = 13 \)

15) \( x^2 - \frac{1}{2}x - \frac{3}{16} = 0 \)

16) \( x^2 + \frac{2}{3}x - \frac{1}{3} = 0 \)

17) \( 3x^2 + x - \frac{1}{2} = 0 \)
18) \( 2x^2 - 3x - 1 = 0 \)

Solve each equation by using the Quadratic Formula.

19) \( x^2 - 4x + 2 = 0 \)

20) \( x^2 + 4x + 2 = 0 \)

21) \( x^2 - 4x - 1 = 0 \)

22) \( x^2 + 6x + 1 = 0 \)

23) \( 2x^2 - 5x + 3 = 0 \)

24) \( 2x^2 + 5x + 3 = 0 \)

25) \( 4y^2 - y + 2 = 0 \)

26) \( 4x^2 + x + 1 = 0 \)

27) \( 4x^2 = 1 - 2x \)
28) \(2x^2 = 1 - 2x\)

29) \(4x^2 = 9x\)

30) \(5x = 4x^2\)
Module 1: Polynomial, Rational, and Radical Relationships

Concept: Polynomial Functions (The Remainder)

Time Frame: 4 days

Common Core State Standard

Understand the relationship between zeros and factors of polynomials

A-APR.239 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number $a$, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

Objectives/Skills

- Evaluate polynomial functions
- Identify general shapes of graphs of polynomial functions
- Find maxima and minima of polynomial functions
- Write expressions in quadratic form
- Use quadratic techniques to solve equations
- Evaluate functions using synthetic division
- Determine whether a binomial is a factor of a polynomial by using synthetic division
- Determine the number and type of roots for a polynomial equation
- Find the zeros of a polynomial function
- Identify the possible rational zeros of a polynomial function
- Find all the rational zeros of a polynomial function

Examples

$$\begin{align*}
\frac{4x^2 + 4}{x^2 - 1} &= \frac{4x^4 + 0x^3 + 0x^2 + 0x + 9}{4x^4 - 4x^2} \\
&= \frac{4x^2 + 0x^2}{4x^2 - 4} \\
&= \frac{4x^2}{4 + 9} = 13 \\
&= 4x^2 + 4 + \frac{13}{x^2 - 1}
\end{align*}$$

Instructional Strategies:

- Review the concepts of traditional long division.
- Modeling Real-World Data p. 359

Resources

Comprehensive Curriculum:

- Unit 5, Activity 12: Remainder and Factor Theorems (pasted below)
- Unit 5, Activity 13: The Calculator and Exact Roots of Polynomial Equations (pasted below)
- Unit 5, Activity 14: The Rational Root Theorem and Solving Polynomial Equations (pasted below)
- Unit 5, Activity 15: Graphing Polynomial Functions (pasted below)

Textbook:

- Section 7-1: Polynomial Function (p. 346)

Henrico:

- Polynomial Functions: http://teachers.henrico.k12.va.us/math/hcpsalgebra2/8-1.htm
- Remainder and Factor Theorems http://teachers.henrico.k12.va.us/math/hcpsalgebra2/8-2.htm
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<tr>
<td>Section 7-4: The Remainder and Factor Theorems (p. 365)</td>
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Activity 12: Remainder and Factor Theorems

Materials List: paper, pencil, graphing calculator, Factor Theorem Discovery Worksheet BLM

In this activity, the students will evaluate a polynomial for a given value of the variable using synthetic division, and they will determine if a given binomial is a factor of a given polynomial.

Math Log Bellringer:
Use long division and synthetic division to simplify the following problem.

\[(2x^3 + 3x^2 - 8) \div (x - 4)\]

Solution: \[2x^2 + 11x + 44 + \frac{80}{x - 4}\]

Activity:

- Factor Theorem Discovery Worksheet:
  - In this worksheet, the students will use synthetic division to find a relationship between the remainder when dividing a polynomial by \((x - c)\) and the value of the polynomial at \(f(c)\) developing the Remainder Theorem. They will use this information to determine when \((x - c)\) is a factor of a polynomial thus developing the Factor Theorem.
  - Distribute the Factor Theorem Discovery Worksheet BLM. This worksheet should be used as guided discovery. Allow students to work in pairs or groups stopping after each section to ascertain understanding.
  - After questions #1 and #2 under the Synthetic Division section, have a student write the answers on the board for others to check.
  - After questions #3 and #4, ask students to complete the Remainder Theorem. It states: If \(P(x)\) is a polynomial and \(c\) is a number, and if \(P(x)\) is divided by \(x - c\), then the remainder equals \(P(c)\).
  - In the beginning of the Factor Theorem section, have students verbalize the definition of factor \(\equiv\) two or more numbers or polynomials that are multiplied together to get a third number or polynomial. Allow the students to complete the problems in this section to develop the Factor Theorem: If \(P(x)\) is a polynomial, then \(x - c\) is a factor of \(P(x)\) if and only if \(P(c) = 0\). Have students define a depressed polynomial. Make sure students understand that the goal of this process is to develop a quadratic depressed equation that can be solved by quadratic function methods, such as the quadratic formula or simple factoring.
  - When the theorems have been developed, have students practice the concepts using the Factor Theorem Practice section of the BLM.

- Assign additional problems from the math textbook if necessary.
**Synthetic Division**

1. \((x^3 + 8x^2 - 5x - 84) \div (x + 5)\)
   
   (a) Use synthetic division to divide and write the answers in equation form as
   \[
   \frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient} + \text{remainder}}{\text{divisor}}
   \]
   \[
   \frac{x^3 + 8x^2 - 5x - 84}{x + 5} =
   \]
   
   (b) Multiply both sides of the equation by the divisor (do not expand) and write in equation form as polynomial = (divisor)(quotient) + remainder) in other words
   \[
P(x) = (x - c)(Q(x)) + \text{Remainder}
   \]
   \[
P(x) =
   \]

2. \((x^3 + 8x^2 - 5x - 84) \div (x - 3)\) (Same directions as #1)
   
   (a) \[
   \frac{x^3 + 8x^2 - 5x - 84}{x - 3} =
   \]
   
   (b) \[
P(x) =
   \]

**Remainder Theorem**

3. What is the remainder in #1b above? _______ What is c? _____ Find \(P(-5)\). _______

4. What is the remainder in #2b above? _______ What is c? _____ Find \(P(3)\). _______

5. Complete the Remainder Theorem: If \(P(x)\) is a polynomial and \(c\) is a number, and if \(P(x)\) is divided by \(x - c\), then _______

6. Use your calculators to verify the Remainder Theorem.
   
   (a) Enter \(P(x) = x^3 + 8x^2 - 5x - 84\) into \(y_1\) and find \(P(-5)\) and \(P(3)\) on the home screen as \(y_1(-5)\) and \(y_1(3)\).
   
   (b) Practice: \(f(x) = 4x^3 - 6x^2 + 2x - 5\). Find \(f(3)\) using synthetic division and verify on the calculator.

   (c) Explain why synthetic division is sometimes called *synthetic substitution*.
Factor Theorem

(7) Define factor \( \equiv \) ____________________________________________________________________________

(8) Factor the following:
   (a) 12  (b) \( x^2 - 9 \)  (c) \( x^2 - 5 \)  (d) \( x^2 + 4 \)
   (e) \( x^3 + 8x^2 - 5x - 84 \) (Hint: See #2b above.) = ____________________________________________________________________________

(9) Using 8(e) complete the Factor Theorem: If \( P(x) \) is a polynomial, then \( x - c \) is a factor of
    \( P(x) \) if and only if ____________________________________________________________________________

(10) Work the following problem to verify the Factor Theorem: Factor \( f(x) = x^2 + 3x + 2 \) and find \( f(-2) \)
    and \( f(-1) \).

(11) In #1 and #2 above you redefined the division problem as \( P(x) = (x - c)(Q(x)) + \text{Remainder} \).
    \( Q(x) \) is called a depressed polynomial because the powers of \( x \) are one less than the powers of \( P(x) \).
    The goal is to develop a quadratic depressed equation that can be solved by quadratic function methods.

(a) In #2b, you rewrote \( \frac{x^3 + 8x^2 - 5x - 84}{x - 3} = x^2 - 11x + 28 \) and \( P(x) = (x^2 + 11x + 28)(x-3) \)
    What is the depressed equation? ____________________________________________________________________________

(b) Finish factoring \( x^3 + 8x^2 - 5x - 84 = \) ____________________________________________________________________________
    List all the zeroes: ____________________________________________________________________________

(12) (a) Use synthetic division to determine if \( (x - 2) \) is a factor of \( y = x^3 + 2x^2 - 5x - 6 \).

   (b) What is the depressed equation? ____________________________________________________________________________

   (c) Factor \( y \) completely: ____________________________________________________________________________
Factor Theorem Practice

Given one factor of the polynomial, use synthetic division and the depressed polynomial to factor completely.

(1a) \( x + 1; x^3 + x^2 - 16x - 16 \), \hspace{1cm} (1b) \( x + 6; x^3 + 7x^2 - 36 \).

Given one factor of the polynomial, use synthetic division to find all the roots of the equation.

(2a) \( x - 1; x^3 - x^2 - 2x + 2 = 0 \), \hspace{1cm} (2b) \( x + 2; x^3 - x^2 - 2x + 8 = 0 \).

Given two factors of the polynomial, use synthetic division and the depressed polynomials to factor completely. (Hint: Use the second factor in the 3rd degree depressed polynomial to get a depressed quadratic polynomial, then factor.)

(3a) \( x - 1, x - 3; x^4 - 10x^3 + 35x^2 - 50x + 24 \) \hspace{1cm} (3b) \( x + 3, x - 4, x^4 - 2x^3 - 13x^2 + 14x + 24 \).
Module 1: Polynomial, Rational, and Radical Relationships

Name_________________________ Key_________________________ Date_________________________

**Synthetic Division**

(1) \((x^3 + 8x^2 - 5x - 84) \div (x + 5)\)

(a) Use synthetic division to divide and write the answers in equation form as

\[
\frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}
\]

\[
\frac{x^3 + 8x^2 - 5x - 84}{x + 5} = x^2 + 3x - 20 + \frac{16}{x + 5}
\]

(b) Multiply both sides of the equation by the divisor (do not expand) and write in equation form as polynomial = (divisor)(quotient) + remainder) in other words

\[P(x) = (x - c)(Q(x)) + \text{Remainder}\]

\[P(x) = (x + 5)(x^2 + 3x - 20) + 16\]

(2) \((x^3 + 8x^2 - 5x - 84) \div (x - 3)\) (Same directions as #1)

(a) \[
\frac{x^3 + 8x^2 - 5x - 84}{x - 3} = x^2 + 11x + 28 + \frac{0}{x - 3}
\]

(b) \[P(x) = (x - 3)(x^2 + 11x + 28) + 0\]

**Remainder Theorem**

(3) What is the remainder in #1b above? ____16____ What is c? ____-5____ Find \(P(-5)\). ____16____.

(4) What is the remainder in #2b above? ____0____ What is c? ____3____ Find \(P(3)\). ____0____.

(5) Complete the Remainder Theorem: If \(P(x)\) is a polynomial and \(c\) is a number, and if \(P(x)\) is divided by \(x - c\), then the remainder equals \(P(c)\).

(6) Use your calculators to verify the Remainder Theorem.

(a) Enter \(P(x) = x^3 + 8x^2 - 5x - 84\) into \(y_1\) and find \(P(-5)\) and \(P(3)\) on the home screen as \(y_1(-5)\) and \(y_1(3)\).

(b) Practice: \(f(x) = 4x^3 - 6x^2 + 2x - 5\). Find \(f(3)\) using synthetic division and verify on the calculator.

\[
\begin{array}{c|cccc}
3 & 4 & -6 & 2 & -5 \\
& & 12 & 18 & 60 \\
\hline
& 4 & 6 & 20 & 55
\end{array}
\]

\[f(3) = 4(3)^3 - 6(3)^2 + 2(3) - 5 = 55\]

(c) Explain why synthetic division is sometimes called synthetic substitution. See 6(b)
**Factor Theorem**

(7) Define factor \(\equiv\) **two or more numbers or polynomials that are multiplied together to get a third number or polynomial.**

(8) Factor the following:
   (a) \(12\)  
   (b) \(x^2 - 9\)  
   (c) \(x^2 - 5\)  
   (d) \(x^2 + 4\)

\(12 = (3)(4)\)
\((x - 3)(x + 3)\)
\((x - \sqrt{5})(x + \sqrt{5})\)
\((x + 2i)(x - 2i)\)

(e) \(x^3 + 8x^2 - 5x - 84\) (Hint: See #2b above.) \(= (x - 3)(x^2 + 11x + 28)\)

(9) Using 8(e) complete the Factor Theorem: If \(P(x)\) is a polynomial, then \(x - c\) is a factor of \(P(x)\) if and only if \(P(c) = 0\). (The remainder is 0 therefore \(P(c)\) must be 0.)

(10) Work the following problem to verify the Factor Theorem: Factor \(f(x) = x^2 + 3x + 2\) and find \(f(-2)\) and \(f(-1)\).

\(x^2 + 3x + 2 = (x + 2)(x + 1)\)
\(f(-2) = 0, f(-1) = 0\)

(11) In #1 and #2 above you redefined the division problem as \(P(x) = (x - c)(Q(x)) + \text{Remainder.}\)

\(Q(x)\) is called a **depressed polynomial** because the powers of \(x\) are one less than the powers of \(P(x)\). The goal is to develop a quadratic depressed equation that can be solved by quadratic function methods.

(a) In #2b, you rewrote \(\frac{x^3 + 8x^2 - 5x - 84}{x - 3} = x^2 + 11x + 28\) and \(P(x) = (x^2 + 11x + 28)(x - 3)\)

What is the depressed equation? \(Q(x) = x^2 - 11x + 28\)

(b) Finish factoring \(x^3 + 8x^2 - 5x - 84 = (x - 3)(x - 7)(x - 4)\)

List all the zeroes: \(\{3, 7, 4\}\)

(12) (a) Use synthetic division to determine if \((x - 2)\) is a factor of \(y = x^3 + 2x^2 - 5x - 6\).

\[
\begin{array}{c|cccc}
2 & 1 & 2 & -5 & -6 \\
\hline & & 2 & 8 & 6 \\
\end{array}
\]

Yes, \((x - 2)\) is a factor.

(b) What is the depressed equation? \(x^2 + 4x + 3\)

(c) Factor \(y\) completely: \((x - 2)(x + 3)(x + 1)\)
Given one factor of the polynomial, use synthetic division and the depressed polynomial to factor completely.

(1a) \( x + 1; \ x^3 + x^2 - 16x - 16 \), \( (x + 1)(x - 4)(x + 4) \)

(1b) \( x + 6; \ x^3 + 7x^2 - 36 \), \( (x + 6)(x + 3)(x - 2) \)

Given one factor of the polynomial, use synthetic division to find all the roots of the equation.

(2a) \( x - 1; \ x^3 - x^2 - 2x + 2 = 0 \), \( \{1, \sqrt{2}, -\sqrt{2}\} \)

(2b) \( x + 2; \ x^3 - x^2 - 2x + 8 = 0 \), \( \{-2, \frac{3}{2} + \frac{\sqrt{7}}{2}i, \frac{3}{2} - \frac{\sqrt{7}}{2}i\} \)

Given two factors of the polynomial, use synthetic division and the depressed polynomials to factor completely. (Hint: Use the second factor in the 3rd degree depressed polynomial to get a depressed quadratic polynomial, then factor.)

(3a) \( x - 1, x - 3; \ x^4 - 10x^3 + 35x^2 - 50x + 24 \), \( (x - 1)(x - 3)(x - 2)(x + 4) \)

(3b) \( x + 3, x - 4, x^4 - 2x^3 - 13x^2 + 14x + 24 \), \( (x + 3)(x - 4)(x - 2)(x + 1) \)
Activity 13: The Calculator and Exact Roots of Polynomial Equations

Materials List: paper, pencil, graphing calculator, Exactly Zero BLM

In this activity, the students will use the calculator and a synthetic division program to help find the exact roots of polynomial equations.

Math Log Bellringer:

Graph \( f(x) = x^3 + 5x^2 - 18 \) on your graphing calculator and find all zeroes. Discuss how you know how many roots and zeroes exist.

*Teacher Note: Students must ZOOM IN around \(-3\) to find both negative zeroes.*

Solution: zeroes: \( \{-3, -3.646, 1.646\} \), The degree of the polynomial tells how many roots there are; but some roots may be imaginary and some may be double roots, so there are at most three different roots and at most three different zeroes.

Activity:

- Use the Bellringer to review the following concepts from Unit 2:
  1. finding zeroes of a polynomial on a graphing calculator
  2. determining the maximum number of roots for a polynomial equation
  3. remembering what a double zero looks like on a graph
  4. approximate values vs exact values

- Have the students decide how to use the integer root they found from the graphs and from synthetic divisions to find the exact answers of the Bellringer problems.

  Solutions: Use the integer root \( x = -3 \) and synthetic division to find the depressed equation which is a quadratic equation. Then use the quadratic formula to find the exact roots

  \( \{ -3, -1 + \sqrt{7}, -1 - \sqrt{7} \} \)

- The problem with using the Factor Theorem is finding one or more of the rational roots to use in synthetic division to create a depressed quadratic equation. The students can find the integer or rational roots found on the calculator and synthetic division to find the irrational or imaginary roots.
  - Have students find the exact roots and factors for the following equation explaining their reasoning: \( x^4 - 6x^3 + 13x^2 - 24x + 36 = 0 \).

  Solutions: From the graph, it is obvious that there is a double root at \( x = 3 \), so 3 would be used twice — once in synthetic division in the original equation and then in the depressed equation to get to a quadratic equation that can be solved.

\[
\begin{array}{c|cccc}
3 & 1 & -6 & 13 & -24 \\
3 & 3 & -9 & 12 & -36 \\
\hline
1 & -3 & 4 & -12 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
3 & 1 & -3 & 4 & -12 \\
3 & 3 & 0 & 12 & \\
\hline
1 & 0 & 4 & 0 & \\
\end{array}
\]
Depressed quadratic equation: \( x^2 + 4 = 0 \Rightarrow x = \pm 2i \)
Roots: \( \{3, 3, \pm 2i\} \), factors: \( (x - 3)^2(x - 2i)(x + 2i) \)

- If the students are going to use the calculator to find the rational roots, then it is logical that they could use the calculator to run a synthetic division program that will generate that depressed equation. This program is available for the TI 83 and 84 at the following website.
  [http://www.ticalc.org/pub/83plus/basic/math/](http://www.ticalc.org/pub/83plus/basic/math/)

- Exactly Zero BLM:
  - On the Exactly Zero BLM, the students will practice finding the exact zeroes by first graphing the function on the calculator to find one or more rational roots and then using these roots in synthetic division (either by hand or using the program). Repeated use of synthetic division will generate a depressed quadratic equation which can then be solved by one of the methods for solving quadratic equations.
  - Distribute the Exactly Zero BLM and allow the students to work in pairs.
  - When students complete the worksheet, check their answers and assign the following problem to be worked individually.
    Find the roots and factors of the following equation:
    \( x^4 - 6x^3 - 2x^2 - 6x + 5 = 0 \)
    Solution: Roots: \( \{1, 1, -2 + i, -2 - i\} \), factors: \( (x - 1)^2(x - 2 + i)(x - 2 - i) \)
Graph the following on your calculator and find all exact zeroes and roots and factors:

(1) \( f(x) = x^3 + 2x^2 - 10x + 4 \)

(2) \( f(x) = x^4 + 2x^3 - 4x^2 - 6x + 3 \)

(3) \( f(x) = x^4 + 8x^3 - 22x^2 - 48x + 96 \)

(4) \( f(x) = 2x^3 + 7x^2 - x - 2 \)  \( \text{Hint: Leading coefficient is 2; therefore, factors must multiply} \)

\[ \text{out to get that coefficient} \]

(5) \( f(x) = 3x^3 - 4x^2 - 28x - 16 \)

(6) Discuss the process used to find the exact answers.
Graph the following on your calculator and find all exact zeroes and roots and factors:

1. \( f(x) = x^3 + 2x^2 - 10x + 4 \)
   - zeroes/roots: \( \{2, 2 + \sqrt{6}, 2 - \sqrt{6}\} \), factors: \( f(x) = (x - 2)(x - 2 - \sqrt{6})(x - 2 + \sqrt{6}) \)

2. \( f(x) = x^4 + 2x^3 - 4x^2 - 6x + 3 \)
   - zeroes/roots: \( \{3, -3, -1 + \sqrt{2}, -1 - \sqrt{2}\} \), factors: \( f(x) = (x - 3)(x + 3)(x + 1 - \sqrt{2})(x + 1 + \sqrt{2}) \)

3. \( f(x) = x^4 + 8x^3 - 22x^2 - 48x + 96 \)
   - zeroes: \( x = -4 \), roots: \( \{-4, -i\sqrt{6}, -i\sqrt{6}\} \), factors: \( f(x) = (x + 4)^2(x - i\sqrt{6})(x + i\sqrt{6}) \)

4. \( f(x) = 2x^3 + 7x^2 - x - 2 \) (Hint: Leading coefficient is 2; therefore, factors must multiply out to get that coefficient)
   - zeroes/roots: \( \left\{ \frac{-1}{2}, \frac{3 + \sqrt{17}}{2}, \frac{3 - \sqrt{17}}{2} \right\} \), factors: \( f(x) = (2x + 1)\left(x - \frac{3 + \sqrt{17}}{2}\right)\left(x - \frac{3 - \sqrt{17}}{2}\right) \)

5. \( f(x) = 3x^3 - 4x^2 - 28x - 16 \)
   - zeroes/roots: \( \left\{ 4, -2, -\frac{2}{3} \right\} \), factors: \( f(x) = (3x - 2)(x + 4)(x + 2) \)

6. Discuss the process used to find the exact answers. Find all rational roots on the calculator. Use these with synthetic division to find a depressed quadratic equation and solve with the quadratic formula.
Activity 14: The Rational Root Theorem and Solving Polynomial Equations

Materials List: paper, pencil, graphing calculator, Rational Roots of Polynomials BLM, Exactly Zero BLM from Activity 13

In this activity, the students will use the Rational Root Theorem and synthetic division to solve polynomial equations.

**Math Log Bellringer:** Distribute the Rational Roots of Polynomials BLM. Have students complete the *vocabulary self awareness* (view literacy strategy descriptions) chart. They should rate their personal understanding of each number system with either a “+” (understands well), a “✓” (limited understanding or unsure), or a “−” (don’t know). They should then look back at the Exactly Zero BLM completed in Activity 13 and list all the roots found and place them in the correct category in the chart. Have students refer back to the chart later in the unit to determine if their personal understanding has improved. For terms in which students continue to have checks and minuses, additional teaching and review may be necessary.

<table>
<thead>
<tr>
<th>Complex Number System Terms</th>
<th>+</th>
<th>✓</th>
<th>–</th>
<th>Root from Exact Zero BLM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 integer</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 rational number</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>3 irrational number</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>4 real number</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>5 imaginary number</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 complex number</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Activity:**

- Use the Bellringer to make sure students can classify types of numbers, a skill begun in Unit 4.
- Rational Roots of Polynomials:
  - The remainder of the Rational Roots of Polynomials BLM should be a teacher guided interactive worksheet.
  - Have students define *rational number*. Possible student answers: (1) a repeating or terminating decimal, (2) a fraction, (3) $\frac{p}{q}$ where $p$ and $q$ are integers and $q \neq 0$.
  - Have students list the rational roots in each of the Exactly Zero BLM problems from Activity 13.
    - What is alike about all the polynomials that have integer rational roots? *Solution: leading coefficient of 1.*
    - What is alike about all the polynomials that have fraction rational roots? *Solution: The leading coefficient is the denominator.*
  - State the Rational Root Theorem: If a polynomial has integral coefficients, then any rational roots will be in the form $\frac{p}{q}$ where $p$ is a factor of the constant and $q$ is a factor of the leading coefficient.
Discuss the following theorems and how they apply to the problems above:

- **Fundamental Theorem of Algebra**: Every polynomial function with complex coefficients has at least one root in the set of complex numbers.
- **Number of Roots Theorem**: Every polynomial function of degree $n$ has exactly $n$ complex roots. (Some may have multiplicity.)
- **Complex Conjugate Root Theorem**: If a complex number $a + bi$ is a solution of a polynomial equation with real coefficients, then the conjugate $a - bi$ is also a solution of the equation.

Have students decide how to choose which of the many rational roots to use to begin synthetic division. Relate back to finding the zeroes on a calculator by entering a lower bound and upper bound.

Discuss continuity of polynomials. Develop the Intermediate Value Theorem for Polynomials: (as applied to locating zeroes). If $f(x)$ defines a polynomial function with real coefficients, and if for real numbers $a$ and $b$ the values of $f(a)$ and $f(b)$ are opposite signs, then there exists at least one real zero between $a$ and $b$.

Have students apply the Rational Root Theorem to solve the last polynomial.

- Assign additional problems in the math textbook for practice.
Module 1: Polynomial, Rational, and Radical Relationships

Name: ___________________________ Date: ___________________________

**Vocabulary Self-Awareness Chart**

(1) Rate your understanding of each number system with either a “+” (understand well), a “✓” (limited understanding or unsure), or a “−” (don’t know)

<table>
<thead>
<tr>
<th>Complex Number System Terms</th>
<th>+</th>
<th>✓</th>
<th>−</th>
<th>Roots from Exact Zero BLM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 integer</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 rational number</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 irrational number</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 real number</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 imaginary number</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 complex number</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(2) List all of the roots found in the Exact Zero BLM completed in Activity #13.

1. \( f(x) = x^3 + 2x^2 - 10x + 4 \)
2. \( f(x) = x^4 + 4x^2 - 6x + 3 \)
3. \( f(x) = x^4 + 8x^3 - 22x^2 - 48x + 96 \)
4. \( f(x) = 2x^3 + 7x^2 - x - 2 \)
5. \( f(x) = 3x^3 - 4x^2 - 28x - 16 \)

(3) Fill the roots in the chart in the proper classification.

**Rational Root Theorem**

(4) Define rational number: ___________________________

__________________________________________

(5) Circle all the rational roots in the equations above.

(6) What is alike about all the polynomials that have integer rational roots?

(7) What is alike about all the polynomials that have fraction rational roots?

(8) Complete the Rational Root Theorem: If a polynomial has integral coefficients, then any rational roots will be in the form \( \frac{p}{q} \) where \( p \) is__________________________

and \( q \) is__________________________

(9) Identify the \( p = \) constant and the \( q = \) leading coefficient of the following equations from the Exact Zero BLM and list all possible rational roots:

____________________________________________

____________________________________________

____________________________________________

____________________________________________

____________________________________________

____________________________________________
Module 1: Polynomial, Rational, and Radical Relationships

### Additional Theorems for Graphing Aids

10. Fundamental Theorem of Algebra: Every polynomial function with complex coefficients has at least one root in the set of complex numbers.

11. Number of Roots Theorem: Every polynomial function of degree \( n \) has exactly \( n \) complex roots. (Some may have multiplicity.)

12. Complex Conjugate Root Theorem: If a complex number \( a + bi \) is a solution of a polynomial equation with real coefficients, then the conjugate \( a - bi \) is also a solution of the equation. (e.g. If \( 2 + 3i \) is a root then \( 2 - 3i \) is a root.)

13. Intermediate Value Theorem for Polynomials: (as applied to locating zeroes). If \( f(x) \) defines a polynomial function with real coefficients, and if for real numbers \( a \) and \( b \) the values of \( f(a) \) and \( f(b) \) are opposite signs, then there exists at least one real zero between \( a \) and \( b \).

(a) Consider the following chart of values for a polynomial. Because a polynomial is continuous, in what intervals of \( x \) does the Intermediate Values Theorem guarantee a zero?

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-1482</td>
<td>-341</td>
<td>216</td>
<td>357</td>
<td>250</td>
<td>63</td>
<td>-36</td>
<td>121</td>
<td>702</td>
<td>1875</td>
</tr>
</tbody>
</table>

This is data for the polynomial \( f(x) = 28x^3 + 68x^2 - 83x - 63 \).

(b) List all the possible rational roots:

(c) Circle the ones that lie in the Interval of Zeroes.

(d) Use synthetic division with the circled possible rational roots to find a depressed equation to locate the remaining roots.
Unit 5, Activity 14, Rational Roots of Polynomials with Answers

Vocabulary Self–Awareness Chart

(1) Rate your understanding of each number system with either a “+” (understand well), a “✓” (limited understanding or unsure), or a “−” (don’t know)

<table>
<thead>
<tr>
<th>Complex Number System Terms</th>
<th>+</th>
<th>✓</th>
<th>−</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 integer</td>
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<tr>
<td>2 rational number</td>
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<tr>
<td>3 irrational number</td>
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<tr>
<td>4 real number</td>
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<tr>
<td>5 imaginary number</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>6 complex number</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Roots from Exact Zero BLM

2, −2, 3, −3, 4, −4

2, −2, 3, −3, 4, −4, −1/2, 2/3

2 + √6, 2 − √6, −1 + √2, −1 − √2, −3/2 + √17/2, −3/2 − √17/2

all the answers in #1 – 3 above

i√6, −i√6

all the answers in #1 – 5 above

(2) List all of the roots found in the Exact Zero BLM completed in Activity #13.

(1) \( f(x) = x^3 + 2x^2 - 10x + 4 \)

\( \{2, 2 + \sqrt{6}, 2 - \sqrt{6}\} \)

(2) \( f(x) = x^4 + 2x^3 - 4x^2 - 6x + 3 \)

\( \{3, -3, -1 + \sqrt{5}, -1 - \sqrt{5}\} \)

(3) \( f(x) = x^4 + 8x^3 - 22x^2 - 48x + 96 \)

\( \{-4, -4, i\sqrt{6}, -i\sqrt{6}\} \)

(4) \( f(x) = 2x^3 + 7x^2 - x - 2 \)

\( \left\{-\frac{1}{2}, -\frac{3}{2} + \frac{\sqrt{17}}{2}, -\frac{3}{2} - \frac{\sqrt{17}}{2}\right\} \)

(5) \( f(x) = 3x^3 - 4x^2 - 28x - 16 \)

\( \left\{4, -2, \frac{2}{3}\right\} \)

(3) Fill the roots in the chart in the proper classification.

Rational Root Theorem

(4) Define rational number: \( \frac{p}{q} \), where \( p \) and \( q \) are integers and \( q \neq 0 \). All terminating and repeating decimals can be expressed as fractions in this form.

(5) Circle all the rational roots in the equations above.

(6) What is alike about all the polynomials that have integer rational roots? The leading coefficient = 1

(7) What is alike about all the polynomials that have fraction rational roots? The leading coefficient ≠ 1

(8) Complete the Rational Root Theorem: If a polynomial has integral coefficients, then any rational roots will be in the form \( \frac{p}{q} \), where \( p \) is a factor of the constant and \( q \) is a factor of the leading coefficient.
(9) Identify the $p = \text{constant}$ and the $q = \text{leading coefficient}$ of the following equations from the Exact Zero BLM and list all possible rational roots:

<table>
<thead>
<tr>
<th>polynomial</th>
<th>factors of $p$</th>
<th>factors of $q$</th>
<th>possible rational roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $f(x) = x^3 + 2x^2 - 10x + 4$</td>
<td>$\pm 1, \neq 2, \pm 4$</td>
<td>$\pm 1$</td>
<td>$\pm 1, \neq 2, \pm 4$</td>
</tr>
<tr>
<td>(2) $f(x) = x^4 + 2x^3 - 4x^2 - 6x + 3$</td>
<td>$\pm 1, \pm 3$</td>
<td>$\pm 1$</td>
<td>$\pm 1, \pm 3$</td>
</tr>
<tr>
<td>(3) $f(x) = x^4 + 8x^3 - 22x^2 - 48x + 96$</td>
<td>$\pm 1, \pm 2, \pm 7, \pm 14, \pm 49, \pm 96$</td>
<td>$\pm 1$</td>
<td>$\pm 1, \pm 2, \pm 7, \pm 14, \pm 49, \pm 96$</td>
</tr>
<tr>
<td>(4) $f(x) = 2x^3 + 7x^2 - x - 2$</td>
<td>$\pm 1, \pm 2$</td>
<td>$\pm 1, \pm 2$</td>
<td>$\pm 1, \pm 2, \pm 2$</td>
</tr>
<tr>
<td>(5) $f(x) = 3x^3 - 4x^2 - 28x - 16$</td>
<td>$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$</td>
<td>$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$</td>
<td>$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$</td>
</tr>
</tbody>
</table>

Additional Theorems for Graphing Aids

(10) Fundamental Theorem of Algebra: Every polynomial function with complex coefficients has at least one root in the set of complex numbers.

(11) Number of Roots Theorem: Every polynomial function of degree $n$ has exactly $n$ complex roots. (Some may have multiplicity.)

(12) Complex Conjugate Root Theorem: If a complex number $a + bi$ is a solution of a polynomial equation with real coefficients, then the conjugate $a - bi$ is also a solution of the equation. (e.g. If $2 + 3i$ is a root then $2 - 3i$ is a root.)

(13) Intermediate Value Theorem for Polynomials: (as applied to locating zeroes). If $f(x)$ defines a polynomial function with real coefficients, and if for real numbers $a$ and $b$ the values of $f(a)$ and $f(b)$ are opposite signs, then there exists at least one real zero between $a$ and $b$.

(a) Consider the following chart of values for a polynomial. Because a polynomial is continuous, in what intervals of $x$ does the Intermediate Values Theorem guarantee a zero?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-5$</th>
<th>$-4$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
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<td>$63$</td>
<td>$-36$</td>
<td>$121$</td>
<td>$702$</td>
<td>$1875$</td>
</tr>
</tbody>
</table>

This is data for the polynomial $f(x) = 28x^3 + 68x^2 - 83x - 63$.

(b) List all the possible rational roots:

- **factors of 63**: $\{\pm 1, \pm 3, \pm 7, \pm 9, \pm 21, \pm 63\}$
- **factors of 28**: $\{\pm 1, \pm 2, \pm 4, \pm 7, \pm 14, \pm 28\}$

possible rational roots:

$\pm \left\{ \frac{1, 3, 7, 9, 21, 63}{2}, \frac{1, 3, 7, 9, 21, 63}{14}, \frac{1, 3, 7, 9, 21, 63}{28} \right\}$

(c) Circle the ones that lie in the interval of Zeros.

\[\left\{ \frac{7, 3, 3, 3}{2}, \frac{7, 3, 3, 3}{4}, \frac{3, 3, 3, 7, 9, 9}{14}, \frac{3, 3, 3, 7, 9, 9}{28}, \frac{3, 3, 3, 7, 9, 9}{4, 4, 7, 14} \right\} \]

Try $\frac{-7}{2}$ first because it is the only one in that interval.

(d) Use synthetic division with the circled possible rational roots to find a depressed equation to locate the remaining roots. $\left\{ \frac{7, 3, 3}{2}, \frac{7, 3, 3}{4} \right\}$
Activity 15: Graphing Polynomial Functions
Materials List: paper, pencil, graphing calculator, Solving the Polynomial Mystery BLM

In this activity, the students will tie together all the properties of polynomial graphs learned in Unit 2 and in the above activities to draw a sketch of a polynomial function with accurate zeroes and end-behavior.

Math Log Bellringer: Graph on your graphing calculator. Adjust WINDOW to see maximum and minimum y values and intercepts. Find exact zeroes and exact roots.

(1) \( f(x) = x^3 - 3x^2 - 5x + 12 \)
(2) \( f(x) = x^4 - 1 \)
(3) \( f(x) = -x^4 + 8x^2 + 9 \)
(4) \( f(x) = -x^3 - 3x \)
(5) Discuss the difference in zeroes and roots

Solutions:
(1) zeroes \(-2, 1.5, 3.5\), roots \(-2, 1.5, 3.5\), (2) zeroes: \(1, -1\), roots: \(1, -1, i, -i\)
(3) zeroes: \(3, -3\), roots: \(3, -3, i, -i\), (4) zeroes: \(0\), roots: \(\pm i\sqrt{3}\)
(5) Zeroes are the x–intercepts on a graph where \(y = 0\). Roots are solutions to a one variable equation and can be real or imaginary.

Activity:

- Use the Bellringer to review the following:
  (1) Unit 2 concepts (end–behavior of odd and even degree polynomials, how end–behavior changes for positive or negative leading coefficients).
  (2) Unit 5 concepts (the Number of Roots Theorem, Rational Root Theorem, and synthetic division to find exact roots).
  (3) What an imaginary root looks like on a graph (i.e. imaginary roots cannot be located on a graph because the graph is the real coordinate system.)  (Students in Algebra II will be able to sketch the general graph with the correct zeroes and end–behavior, but the particular shape will be left to Calculus.)

- Before assigning the problem of graphing a polynomial with all of its properties, ask the students to write a GIST (view literacy strategy descriptions).
  GISTing is an excellent strategy for helping students paraphrase and summarize essential information. Students are required to limit the GIST of a concept to a set number of words. Begin by reminding students of the fundamental characteristics of a summary or GIST by placing these on the board or overhead:
  (1) Shorter than the original text
  (2) A paraphrase of the author’s words and descriptions
  (3) Focused on the main points or events
Assign the following GIST: When you read a mystery, you look for clues to solve the case. Think of solving for the roots of a polynomial equation as a mystery. Discuss all the clues you would look for to find the roots of the equation. Your discussion should be bulleted, concise statements, not full sentences, and cover about ½ sheet of paper.

When students have finished their GISTS, create a list on the board of characteristics that should be examined in graphing a polynomial.

- **Solving the Polynomial Mystery:**
  - In the Solving the Polynomial Mystery BLM, the students will combine all the concepts developed in this unit that help to find the roots of a higher degree polynomial and will check to see if their GISTing was complete.
  - Distribute the Solving the Polynomial Mystery BLM. This is a noncalculator worksheet. Allow students to work in pairs circulating to make sure they are applying all the theorems correctly.
  - When students have completed the graph have them check it on their graphing calculators finding both the graph and the decimal approximations of the roots. Make sure all the elements in the worksheet – intercepts, roots, end-behavior, and ordered pairs in the chart – are located on the graph. (They will not be able to find the maximum and minimum points by hand until Calculus.)
  - Have students return to their GISTS and add any concepts they had forgotten.
Module 1: Polynomial, Rational, and Radical Relationships

Answer #1 – 8 below concerning this polynomial:

\[ f(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3 \]

(1) How many roots does the Fundamental Theorem of Algebra guarantee this equation has? ___

(2) How many roots does the Number of Roots Theorem say this equation has? ______

(3) List all the possible rational roots: ____________________________________________

(4) Use the chart below and the Intermediate Value Theorem to locate the interval/s of the zeroes.

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-(\frac{3}{2})</th>
<th>-1</th>
<th>-(\frac{1}{2})</th>
<th>0</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>(\frac{3}{2})</th>
<th>2</th>
<th>(\frac{5}{2})</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>25</td>
<td>-12</td>
<td>-18</td>
<td>-11</td>
<td>-3</td>
<td>0</td>
<td>-2</td>
<td>-3</td>
<td>9</td>
<td>52</td>
<td>150</td>
</tr>
</tbody>
</table>
(5) If you have one root, use synthetic division to find the depressed equation and rewrite $y$ as a factored equation with one binomial root and the depressed equation.

$$y = (\underline{\phantom{0}}) \ (\underline{\phantom{0}})$$

factor depressed equation

(6) Use synthetic division on the depressed equation to find all the other roots.

List all the roots repeating any roots that have multiplicity. {\underline{\phantom{0}}}

(7) Write the equation factored with no fractions and no exponents greater than one.

$$y = (\underline{\phantom{0}})(\underline{\phantom{0}})(\underline{\phantom{0}})(\underline{\phantom{0}})(\underline{\phantom{0}})$$
(8) Graph $f(x)$ without a calculator using all the available information in questions #1–7 on the previous page.
Answer all the questions on this page concerning this polynomial:

\[ f(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3 \]

(1) How many roots does the Fundamental Theorem of Algebra guarantee this equation has? __1__

(2) How many roots does the Number of Roots Theorem say this equation has? _4_

(3) List all the possible rational roots:

\[ \left\{ \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4} \right\} \]

\[ p = 3 \quad \left\{ \pm 1, \pm 3 \right\} \]
\[ q = 4 \quad \left\{ \pm 1, \pm 2, \pm 4 \right\} \]

(4) Use the chart at below and the Intermediate Value Theorem to locate the interval/s of the zeroes.

\[ (\frac{-3}{2}, 2), \text{ root at } x = \frac{1}{2}, (\frac{3}{2}, 2) \]

\[
\begin{array}{cccccccc}
 x & -2 & -\frac{3}{2} & -1 & \frac{-1}{2} & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 \\
 y & 25 & -12 & -18 & -3 & 0 & -2 & -3 & 9 & 52 & 150 \\
\end{array}
\]

(5) If you have one root, use synthetic division to find the depressed equation and rewrite y as a factored equation with one binomial root and the depressed equation.

\[ y = (x - \frac{1}{2}) \left( 4x^3 - 2x^2 - 12x + 6 \right) \]

\[
\begin{array}{c|cccc}
 \frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 \hline
 & 2 & -1 & -6 & 3 \\
 \hline
 & 4 & -2 & -12 & 6 & 0 \\
\end{array}
\]

\( \frac{1}{2} \) may be a double root so try it again.

(6) Use synthetic division on the depressed equation to find all the other roots.

List all the roots repeating any roots that have multiplicity. \( \left\{ \frac{1}{2}, \frac{1}{2}, \sqrt{3}, -\sqrt{3} \right\} \)

\[
\begin{array}{c|cccc}
 \frac{1}{2} & 4 & -2 & -12 & 6 \\
 \hline
 & 2 & 0 & -6 \\
 \hline
 & 4 & 0 & -12 & 0 \\
\end{array}
\]

(7) Write the equation factored with no fractions and no exponents greater than one.

\[ y = (2x - 1)(2x - 1)(x - \sqrt{3})(x + \sqrt{3}) \]
(8) Graph \( f(x) \) without a calculator using all the available information in questions #1–7 on the previous page.
Concept: Multiplying and Dividing Rational Expressions  

Common Core State Standard  

A-REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Objectives/Skills  
- Simplify rational expressions
- Factor quadratic expressions into trinomials, perfect squares, difference of two squares, and polynomials

Examples

\[
\begin{align*}
\text{Simplify } \frac{2x(x - 5)}{(x - 5)(x^2 - 1)^3} & \quad \text{Simplify } \frac{z^2w - z^2}{z^3 - z^2w}. \\
\quad \text{Simplify each expression.} & \quad a. \quad \frac{4a}{5} \quad \frac{15b^2}{16a^3} \quad \frac{r^2 - 25a^3}{5a - r}.
\end{align*}
\]

Instructional Strategies  
- Foldables Study Organizer p 471

Resources  

Comprehensive Curriculum:
- Unit 3 Activity 1: Simplifying Rational Expressions
- Unit 3 Activity 2: Multiplying and Dividing Rational Expressions

Textbook:
- Section 9-1: Multiplying and Dividing Rational Expressions p.472

Websites:  

Henrico:  

Remediation:  

Videos:
- [www.mathtv.com](http://www.mathtv.com)

Educational Games:  

Practice Problem:  
- [http://spot.pcc.edu/~kkling/Mth_95/SectionIII_Rational_Expressions_Equations_and_Functions/Module4/Module4_Complex_Rational_Expressions.pdf](http://spot.pcc.edu/~kkling/Mth_95/SectionIII_Rational_Expressions_Equations_and_Functions/Module4/Module4_Complex_Rational_Expressions.pdf)

Enrichment:  
- DE Streaming Video Titled: Multiplying and Dividing Rational Expressions—Personal Trainer
Activity 1: Simplifying Rational Expressions

Materials List: paper, pencil, graphing calculator, Math Log Bellringer BLM, Simplifying Rational Expressions BLM

In this activity, the students will review non-positive exponents and use their factoring skills from the previous unit to simplify rational expressions.

Math Log Bellringer: Simplify:

(1) \((x^2)(x^5)\), (2) \((x^2y^5)^4\), (3) \(\frac{x^5}{x^8}\), (4) \(\frac{x^7}{x^3}\), (5) \(\frac{x^3}{x^5}\)

(6) Choose one problem above and write in a sentence the Law of Exponents used to determine the solution.

Solutions:

(1) \(x^7\), Law of Exponents: When you multiply variables with exponents, you add the exponents.

(2) \(x^{20}\), Law of Exponents: When you raise a variable with an exponent to a power, you multiply exponents.

(3) \(x^7, x \neq 0\), Law of Exponents: Same as #2 plus when you divide variables with exponents, you subtract the exponents.

(4) \(1, x \neq 0\), Law of Exponents: Same as #3 plus any variable to the 0 power equals 1.

(5) \(x^{-2} = \frac{1}{x^2}, x \neq 0\), Law of Exponents: Same as #3 plus a variable to a negative exponent moves to the denominator.

(6) See Laws of Exponents above.

Activity:

- Overview of the Math Log Bellringers:
  - As in previous units, each in-class activity in Unit 3 is started with an activity called a Math Log Bellringer that either reviews past concepts to check for understanding (reflective thinking about what was learned in previous classes or previous courses) or sets the stage for an upcoming concept (predictive thinking for that day’s lesson).
  - A math log is a form of a learning log [view literacy strategy descriptions] that students keep in order to record ideas, questions, reactions, and new understandings. Documenting ideas in a log about content being studied forces students to “put into words” what they know or do not know. This process offers a reflection of understanding that can lead to further study and alternative learning paths. It combines writing and reading with content learning. The Math Log Bellringers will include mathematics done symbolically, graphically, and verbally.
  - Since Bellringers are relatively short, blackline masters have not been created for each of them. Write them on the board before students enter class, paste them into an enlarged Word® document or PowerPoint® slide, and project using a TV or digital projector, or print and display using a document or overhead projector. A sample enlarged Math Log Bellringer Word® document has been included in the blackline masters. This sample is the Math Log Bellringer for this activity.
➢ Have the students write the Math Log Bellringers in their notebooks preceding the upcoming lesson during beginning—of—class record keeping and then circulate to give individual attention to students who are weak in that area.

➢ It is important for future mathematics courses that students find denominator restrictions throughout this unit. They should never write $\frac{x}{x} = 1$ unless they also write $x \neq 0$ because the two graphs of these functions are not equivalent.

➢ Write the verbal rules the students created in Bellringer #6 on the board or overhead projector. Use these rules and the Bellringer problems to review Laws of Exponents and develop the meaning of zero and negative exponents. The rules in their words should include the following:

1. “When you multiply like variables with exponents, you add the exponents.”
2. “When you raise a variable with an exponent to a power, you multiply exponents.”
3. “When you divide like variables with exponents, you subtract the exponents.”
4. “Any variable raised to the zero power equals 1.”
5. “A variable raised to a negative exponent moves the variable to the denominator and means reciprocal.”

➢ Simplifying Rational Expressions:

➢ Distribute the Simplifying Rational Expressions BLM. This is a guided discovery/review in which students work only one section at a time and draw conclusions.

➢ Connect negative exponents to what they have already learned about scientific notation in Algebra I and science. To reinforce the equivalencies, have students enter the problems in Section I of the Simplifying Rational Expressions BLM in their calculators. This can be done by getting decimal representations or using the TEST feature of the calculator: Enter $2^{-3} = \frac{1}{2^3}$ (The “=” sign is found under $2^\text{nd}$, TEST (above the MATH). If the calculator returns a “1” then the statement is true; if it returns a “0” then the statement is false.

➢ Use guided practice with problems in Section II of the Simplifying Rational Expressions BLM which students simplify and write answers with only positive exponents.

➢ Have students define rational number to review the definition as the quotient of two integers $\frac{p}{q}$ in which $q \neq 0$, and then define rational algebraic expression as the quotient of two polynomials $P(x)$ and $Q(x)$ in which $Q(x) \neq 0$. Discuss the restrictions on the denominator and have students find the denominator restrictions Section III of the Simplifying Rational Expressions BLM.

➢ Have students simplify $\frac{24}{40}$ in Section IV and let one student explain the steps he/she used.

Make sure there is a discussion of dividing out and cancelling of a common factor. Then have students apply this concept to simplify the expressions in Section V of the Simplifying Rational Expressions BLM.

➢ Remind students that denominator restrictions apply to the original problem, not the simplified problem. To stress this point, have students work Section VI of the Simplifying Rational Expressions BLM.

➢ Conclude the worksheet by having students work the application problem.
Simplify

(1) \((x^2)(x^5)\)

(2) \((x^2y^5)^4\)

(3) \(\frac{(x^5)^3}{x^8}\)

(4) \(\frac{x^7}{x^7}\)

(5) \(\frac{x^3}{x^5}\)
### Laws of Exponents

**I.** Enter the following in your calculators on the home screen:

1. \( 3^0 = \) ___
2. \( 2^{-2} \) and \( \frac{1}{2} = \) ___
3. \( 0.001 \) and \( \frac{1}{10} = \) ___
4. \( 0.0037 \) and \( 3.7 \times 10^{-4} = \) ___

**II.** Simplify and write answers with only positive exponents.

1. \( (x^{-3})(x^4) = \) ___
2. \( (x^{-2})^3 = \) ___
3. \( \frac{x^{-3}}{x^{-2}} = \) ___

**III.** Define rational number as the quotient of two integers in which \( q \neq 0 \) and define rational algebraic expression as the quotient of two polynomials \( P(x) \) and \( Q(x) \) in which \( Q(x) \neq 0 \). Find the denominator restrictions on the following rational expressions.

1. \( \frac{4x^3}{7t} = \) ___
2. \( \frac{3x + 5}{y - 3} = \) ___
3. \( \frac{2x + 5}{3x - 7} = \) ___
4. \( \frac{3x + 2}{x^2 - 5x + 6} = \) ___
5. \( \frac{4}{x^2 - 9} = \) ___

**IV.** Simplify \( \frac{24}{40} \) and explain the steps you used.

**V.** Apply this concept to simplify the following expressions and develop the process to simplify rational expressions. Specify all denominator restrictions.

*(Remember the denominator restrictions apply to the original problem, not the simplified problem.)*

1. \( \frac{-27x^2y^4}{9x^4y} = \) ___
2. \( \frac{a - b}{b - a} = \) ___
3. \( \frac{8 - 2x}{x^2 - 5x + 6} = \) ___
4. \( 8x(4x - 28)^{-1} = \) ___

**VI.** To verify that the denominator restrictions apply to the original problem, not the simplified problem, complete the following:

1. Simplify \( f(x) = \frac{x^3 - 2x^2 + 4x - 8}{x - 2} = \) ___
2. Graph both the original and the simplified form on the graphing calculator. Trace to \( x = 2 \) on both to find \( f(2) \) and \( g(2) \). There is a hole in one graph and not in the other; therefore, they are only equal for all values of \( x \) except \( x = 2 \). Verify this in a table: go to \( 2ND \) \[ TBL SET \], (above \[ WINDOW \]) and TblStart = 0 and go up by increments ( \( \Delta Tbl \) ) = 0.2. Again you will see no value for \( x = 2 \).

**Application**

The side of a regular hexagon is \( 2ab^3 \) and the side of a regular triangle is \( 3a^2b \). Find the ratio of the perimeter of the hexagon to the perimeter of the triangle. Show all your work:
Laws of Exponents

I. Enter the following in your calculators on the home screen:

1. \(3^0 = 1\)
2. \(2^{-3} = \frac{1}{8}\)
3. \(0.001\)
4. \(3.7 \times 10^{-4} = 0.00037\)

II. Simplify and write answers with only positive exponents.

1. \(\left(x^{-3}\right)\left(x^4\right) = x\)
2. \(\left(x^{-2}\right)^3 = \frac{1}{x^6}\)
3. \(\frac{x^3}{x^{-2}} = x^5\)

III. Define rational number as the quotient of two integers \(\frac{p}{q}\) in which \(q \neq 0\) and define rational algebraic expression as the quotient of two polynomials \(P(x), Q(x)\) in which \(Q(x) \neq 0\). Find the denominator restrictions on the following rational expressions.

1. \(t^7 x^4, t \neq 0\)
2. \(a^3 - b^3, a \neq b\)
3. \(x^2 - 5x + 6, x \neq 2, x \neq 3\)
4. \(28x + 7, x \neq 7\)

IV. Simplify \(\frac{24}{40}\) and explain the steps you used.

\[\frac{24}{40} = \frac{3}{5}\]

Use the identity element of multiplication.

V. Apply this concept to simplify the following expressions and develop the process to simplify rational expressions. Specify all denominator restrictions.

(Remember the denominator restrictions apply to the original problem, not the simplified problem.)

1. \(\frac{-27x^2y^4}{9x^4y} = \frac{-3y^3}{x^2}, x \neq 0, y \neq 0\)
2. \(\frac{a-b}{b-a} = -1, a \neq b\)
3. \(\frac{8-2x}{x^2-5x+6} = \frac{-4}{x-3}, x \neq 2, x \neq 3\)
4. \(8x(4x-28)^{-1} = \frac{2x}{x-7}, x \neq 7\)

VI. To verify that the denominator restrictions apply to the original problem, not the simplified problem, complete the following:

1. Simplify \(f(x) = \frac{x^3-2x^2+4x-8}{x-2}\)

2. Graph both the original and the simplified form on the graphing calculator. Trace to \(x = 2\) on both to find \(f(2)\) and \(g(2)\). There is a hole in one graph and not in the other; therefore, they are only equal for all values of \(x\) except \(x = 2\). Verify this in a table: go to \(\text{2ND}, [\text{TBL SET}]\), (above \(\text{WINDOW}\)) and \(\text{TblStart} = 0\) and go up by increments \((\Delta \text{Tbl}) = 0.2\). Again you will see no value for \(x = 2\).

Application

The side of a regular hexagon is \(2ab^3\) and the side of a regular triangle is \(3a^2b\). Find the ratio of the perimeter of the hexagon to the perimeter of the triangle. Show all your work:

\[\frac{4b^2}{3a}\]
Activity 2: Multiplying and Dividing Rational Expressions

Materials List: paper, pencil

In this activity, the students will multiply and divide rational expressions and use their factoring skills to simplify the answer. They will also express domain restrictions.

Math Log Bellringer:

Simplify the following:

1. \( \frac{3}{4} \cdot \frac{10}{11} \)
2. \( \frac{7}{8} \cdot 4 \)
3. \( \frac{4x^2 \cdot y^3}{5y \cdot 12x^5} \)
4. \( \frac{x + 2}{x - 3} \cdot \frac{4}{5} \)
5. \( \frac{2x + 3}{x - 5} \cdot (x - 2) \)
6. \( \frac{x - 2}{x + 4} \cdot \frac{x + 3}{x - 5} \)
7. Write in a sentence the rule for multiplying and simplifying fractions.
8. What mathematical rule allows you to cancel constants?
9. What restrictions should you state when you cancel variables?

Solutions:

1. \( \frac{15}{22} \), (2) \( \frac{7}{2} \), (3) \( \frac{y^2}{15x^3} \), \( y \neq 0 \), (4) \( \frac{4x + 8}{5x - 15} \),
5. \( \frac{2x^2 - x - 6}{x - 5} \), (6) \( \frac{x^2 + x - 6}{x^2 - x - 20} \)

(7) When you multiply fractions, you multiply the numerators and multiply the denominators. Then you find any common factors in the numerator and denominator and cancel them to simplify the fractions.

(8) If “a” is a constant, \( \frac{a}{a} = 1 \), the identity element of multiplication; therefore, you can cancel common factors without changing the value of the expression.

(9) If you cancel variables, you must state the denominator restrictions of the cancelled factor or the expressions are not the same.
Activity:

- Use the Bellringer to review the process of multiplying numerical fractions and have students extend the process to multiplying rational expressions. Students should simplify and state denominator restrictions.

- Have students multiply and simplify \( \frac{x^2 - 4}{x + 3} \cdot \frac{2x + 6}{x^2 + 7x + 10} \) and let students that have different processes show their work on the board. Examining all the processes, have students choose the most efficient (factoring, canceling, and then multiplying). Make sure to include denominator restrictions.

- Have the students work the following \( \frac{3}{4} \div \frac{10}{11} = \) and \( \frac{7}{8} \div 4 \). Define reciprocal and have students rework the Bellringers with a division sign instead of multiplication.

- Application:

  Density is mass divided by volume. The density of solid brass is \( \frac{x + 5}{2} \) g/cm\(^3\). If a sample of an unknown metal in a laboratory experiment has a mass of \( \frac{x^2 + 2x - 15}{2x - 8} \) g and a volume of \( \frac{x^2 + x - 12}{x^2 - 16} \) cm\(^3\), determine if the sample is solid brass.

  Solution: yes

- Critical Thinking Writing Assessment: (See Activity-Specific Assessments at end of unit.)
### Concept: Adding and Subtracting Rational Expressions

<table>
<thead>
<tr>
<th>Common Core State Standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.</td>
</tr>
</tbody>
</table>

### Objectives/Skills
- Determine the LCD
- Add and subtract rational expressions

### Examples:

- **Simplify the following:** \( \frac{5x - 1}{x + 8} - \frac{3x + 4}{x + 8} \)
- **Simplify the following:** \( \frac{4x}{2x - 1} - \frac{5}{x - 6} \)
- **Simplify the following:** \( \frac{1}{x + 1} + \frac{x}{x - 6} - \frac{5x - 2}{x^2 - 5x - 6} \)

### Instructional Strategies
- Foldables Study Organizer p 471

### Resources

#### Comprehensive Curriculum:
- Unit 3 Activity 3: Adding and Subtracting Rational Expressions
- Unit 3 Activity 4: Complex Rational Expressions

#### Textbook:
- Section 9-2: Adding and Subtracting Rational Expressions p. 479

#### Websites:
- Promethean:

- Henrico:
  - [http://teachers.henrico.k12.va.us/math/hcps](http://teachers.henrico.k12.va.us/math/hcps)

### Remediation:

#### Videos:
- [www.mathtv.com](http://www.mathtv.com)

#### Educational Games:

#### Practice Problems:
- [http://cnx.org/content/m21936/latest/](http://cnx.org/content/m21936/latest/)

#### Enrichment:
- DE Streaming Video Titled: Adding and Subtracting
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<th>algebra2/9-4.htm</th>
<th>Rational Expressions--Workouts</th>
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<td>• <a href="http://exchange.smarttech.com/details.html?id=da639207-ffc5-4ae8-9d9a-e5f4e8f939b8">link</a></td>
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</tbody>
</table>
Activity 3: Adding and Subtracting Rational Expressions

Materials List: paper, pencil, Adding & Subtracting Rational Expressions BLM

In this activity, the students will find common denominators to add and subtract rational expressions.

Math Log Bellringer:
Simplify and express answer as an improper fraction:

1. \( \frac{2}{5} + \frac{7}{11} \)
2. \( \frac{2}{15} + \frac{7}{25} \)
3. \( \frac{2}{5} + 6 \)
(4) Write the mathematical process used to add fractions.

Solutions: (1) \( \frac{57}{55} \), (2) \( \frac{31}{75} \), (3) \( \frac{32}{5} \), (4) When you add fractions you have to find a common denominator. To find the least common denominator, use the highest degree of each factor in the denominator.

Activity:

- Use the Bellringer to review the rules for adding and subtracting fractions and relate them to rational expressions.
- Adding/Subtracting Rational Functions BLM:
  - Distribute the Adding & Subtracting Rational Expressions BLM and have students work in pairs to complete. On this worksheet, the students will apply the rules they know about adding and subtracting fractions to adding and subtracting rational expressions with variables.
  - In Section I, have the students write the rule developed from the Bellringers then apply the rule to solve the problems in Section II. Have two of the groups write the problems on the board and explain the process they used.
  - Have the groups work Section III and IV and again have two of the groups write the problems on the board and explain the process they used.
  - Have students work the application problem and one of the groups explain it on the board.
  - Finish by giving the students additional problems adding and subtracting rational expressions from the math textbook.

- Critical Thinking Writing Assessment: (See Activity-Specific Assessments at end of unit.)
Adding/Subtracting Rational Expressions

I. State the Rule for Adding/Subtracting Rational Expressions: ____________________________________________________________

II. Apply this process to find the sums:

(1) \( \frac{x-5}{6x^2} + \frac{2x+6}{6x^2} \)

(2) \( \frac{x-5}{6x^2-54} + \frac{2x+6}{x-3} \)

III. Subtract and simplify.

(3) \( \frac{2}{15} - \frac{7}{25} \)

(4) \( \frac{2}{5} - 6 \)

State the Rule:

IV. Apply this process to find the differences:

(5) \( \frac{x-5}{6x^2} - \frac{2x+6}{6x^2} \)

(6) \( \frac{x-5}{6x^2-54} - \frac{2x+6}{x-3} \)

Application

The time it takes a boat to go downstream is represented by the function \( d(x) = \frac{2}{x+1} \) hours, where \( x \) represents the number of miles. The time it takes a boat to go upstream is represented by the function \( u(x) = \frac{3}{x-1} \) hours.

a. How long in minutes does it take to go 2 miles upstream? 2 miles downstream? Explain why it would be different?

b. Find a rational function \( f(x) \) for the total time in minutes. Then find the total time it takes to go a total of 2 miles upstream then back to the starting point.

c. Find a rational function \( g(x) \) for how much more time it takes to go upstream than downstream.

d. Find how much more time in minutes it takes to go upstream than downstream if you have traveled 2 miles upstream and back to the starting point.
Adding/Subtracting Rational Expressions

I. State the Rule for Adding/Subtracting Rational Expressions: Find the LCD and add the numerators and keep the denominator.

II. Apply this process to find the sums:

1. \( \frac{x - 5}{6x^2} + \frac{2x + 6}{6x^2} = \frac{3x + 1}{6x^2} \)

2. \( \frac{x - 5}{6x^2 - 54} + \frac{2x + 6}{x - 3} = \frac{12x^2 + 73x + 103}{6x^2 - 54} \)

III. Subtract and simplify.

3. \( \frac{2}{15} - \frac{7}{25} = -\frac{11}{75} \)

   State the Rule: Find the LCD and subtract the numerators and keep the denominator.

IV. Apply this process to find the differences:

5. \( \frac{x - 5}{6x^2 - 54} - \frac{2x + 6}{x - 3} = \frac{-2x - 11}{6x^2} \)

6. \( \frac{x - 5}{6x^2 - 54} - \frac{2x + 6}{x - 3} = \frac{-12x^2 - 71x - 113}{6x^2 - 54} \)

Application

The time it takes a boat to go downstream is represented by the function \( d(x) = \frac{2}{x + 1} \) hours, where \( x \) represents the number of miles. The time it takes a boat to go upstream is represented by the function \( u(x) = \frac{3}{x - 1} \) hours.

a. How long in minutes does it take to go 2 miles upstream? 2 miles downstream? Explain why it would be different? \( u(2) = 3 \text{ hours} = 180 \text{ minutes}, d(2) = 40 \text{ minutes}, \text{current helps going downstream} \)

b. Find a rational function \( f(x) \) for the total time in minutes. Then find the total time it takes to go a total of 2 miles upstream then back to the starting point. \( u(x) + d(x) = f(x) = \frac{5x + 1}{x^2 - 1} \) \( f(2) = 220 \text{ minutes} \)

c. Find a rational function \( g(x) \) for how much more time it takes to go upstream than downstream.

\( u(x) - d(x) = g(x) = \frac{x + 5}{x^2 - 1} \)

d. Find how much more time in minutes it takes to go upstream than downstream if you have traveled 2 miles upstream and back to the starting point. \( g(2) = 140 \text{ minutes} \)
Activity 4: Complex Rational Expressions
Materials List: paper, pencil

In this activity, the students will simplify complex rational fractions.

Math Log Bellringer: Multiply and simplify the following:

(1) \(6x^2y^2 \left( \frac{x}{6y^2} + \frac{3y}{2x^2} \right)\)

(2) \((x+2)(x-5)\left( \frac{3}{x+2} + \frac{7}{x-5} \right)\)

(3) What mathematical properties are used to solve the above problems?

Solutions: (1) \(y^3 + 9y\), (2) 10x – 1, (3) First, you use the Distributive Property of Multiplication over Addition. Second, you cancel like factors which use the identity element of multiplication. Then you combine like terms.

Activity:

- Use the Bellringer to review the Distributive Property.
- Define complex fraction and ask students how to simplify \(\frac{1}{6 \cdot 9}\). Most students will invert and multiply.

Discuss an alternate process of multiplying by 18/18 or the LCD ratio equivalent to 1.

- Define complex rational expression and have students determine the best way to simplify \(\frac{1}{x+4} \cdot \frac{y}{5} \cdot \frac{3}{y}\).

Discuss why it would be wrong to work this problem this way: \(\left( \frac{1}{x+4} \right) \left( \frac{1+y}{3} \right)\).

- Have students determine the process to simplify \(\frac{2}{x+3} + \frac{5x}{x^2-9}\).

Solution: \(\frac{7x-6}{6x-6}\)

- Use the math textbook for additional problems.
**Concept: Solving Rational Equations**

**Time Frame: 3 days**

**Common Core State Standard**

Understand solving equations as a process of reasoning and explain the reasoning

A-REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable

A-REI.4 Solve quadratic equations in one variable.

c. Solve quadratic equations by inspection (e.g., for \( x^2 = 49 \)), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as \( a \pm bi \) for real numbers \( a \) and \( b \).

**Objectives/Skills**

- Solve rational equations
- Solve rate problems that are expressed as rational equations

**Examples**

Solve the following equation:

\[
\frac{3}{x+2} - \frac{1}{x} = \frac{1}{5x}
\]

Solve the following equation:

\[
\frac{10}{x(x-2)} + \frac{4}{x} = \frac{5}{x-2}
\]

**Resources**

**Comprehensive Curriculum:**
- Unit 3 Activity 5: Solving Rational Equations
- Unit 3 Activity 6: Applications Involving Rational Expressions

**Textbook:**
- Section 9-6: Solving Rational Equations and Inequalities p. 505

**Websites:**
- Promethean

**Remediation:**

**Videos:**
- [www.mathtv.com](http://www.mathtv.com)

**Practice Problems:**

**Enrichment:**
- DE Streaming Video Titled: Rational expressions and equations
- binomials-in-the-denominator

Henrico:
Activity 5: Solving Rational Equations
Materials List: paper, pencil

In this activity, the students will solve rational equations.

Math Log Bellringer:

Solve: \( \frac{x}{2} + \frac{3x}{4} = 5 \) Is this a rational equation or linear equation? Why?

Solution: \( x = 4 \), This is a linear equation because \( x \) is raised to the first power and there are no variables in the denominators.

Activity:

- Use the Bellringer to discuss alternate ways to solve this linear equation:
  1. finding the LCD and adding fractions.
  2. multiplying both sides of the equation by the LCD to remove fractions, then solve for \( x \). Always check the solution because the answer may be an extraneous root, meaning it is a solution to the transformed equation but not the original equation because of the denominator restrictions.

- Have students solve and check the following:
  1. \( \frac{1}{4x} - \frac{3}{4} = \frac{7}{x} \)
  2. \( \frac{x}{x-2} = \frac{1}{2} + \frac{2}{x-2} \)

Solutions: (1) \( x = -9 \), (2) no solution, 2 is an extraneous root

- Use the following solutions to develop the concept of zeroes of the function. Find the denominator restrictions and the solutions for the following:
  1. \( \frac{x-2}{x+3} = 0 \)
  2. \( \frac{x^2 - x + 12}{3x^2} = 0 \)
  3. \( \frac{x^2 - 6x + 5}{x^2 - 3x - 10} = 0 \)

Solutions: (1) \( x = 2 \), \( x ≠ -3 \), (2) \( x = -4 \), \( x = 4 \), \( x ≠ 0 \), (3) \( x = 1 \), \( x ≠ 5 \), \( x ≠ -2 \)

(4) Write the process you used to find the zeroes.

Solutions:
  1. \( x = 2 \), \( x ≠ -3 \), (2) \( x = -4 \), \( x = 4 \), \( x ≠ 0 \), (3) \( x = 1 \), \( x ≠ 5 \), \( x ≠ -2 \)

(4) To find zeroes of a rational function, cancel common factors in the numerator and denominator, set the numerator equal to zero, and solve for \( x \).
Application:
Every camera lens has a characteristic measurement called focal length, $F$. When the object is in focus, its distance, $D$, from the lens to the subject and the distance, $L$, from the lens to the film, satisfies the following equation. \[
\frac{1}{L} + \frac{1}{D} = \frac{1}{F}.
\] If the distance from the lens to an object is 60 cm and the distance from the lens to the film is 3 cm greater than the focal length, what is the focal length of the lens? Draw a picture of the subject, the film, and the lens and write the variables on the picture. Set up the equation and solve. Discuss the properties used and if the answers are feasible and why.

Solution:
\[
\frac{1}{L} + \frac{1}{D} = \frac{1}{F},
\] $F = 12 \text{ cm}$
Activity 6: Applications Involving Rational Expressions

Materials List: paper, pencil, graphing calculator, Rational Expressions Applications BLM

In this activity, students will solve rate problems that are expressed as rational equations.

Math Log Bellringer:
In an Algebra II class, 2 out of 5 of the students are wearing blue. If 14 of the students are wearing blue, how many are there in the class? Set up a rational equation and solve. Describe the process used.

\[ \frac{2}{5} = \frac{14}{x}, \quad x = 35 \]

Activity:

- Use the Bellringer to review the meaning of ratio (part to part) and proportion (part to whole). Ask the students what is the rate of blue wearers to any color wearers, and have them define rate as a comparison of two quantities with different units. Define proportion as an equation setting two rates equal to each other (with the units expressed in the same order).

- Rational Expressions Applications BLM:
  - On this worksheet the students will set up rational equations, using the concepts of rate and proportion, and solve.
  - Distribute the Rational Expressions Applications BLM and have students work with a partner to setup and solve the application problems. Stop after each problem to check for understanding and to discuss the process used.

- Give additional problems in the math textbook for practice.
Application Problems

1. John’s car uses 18 gallons to travel 300 miles. He has 7 gallons of gas in the car and wants to know how much more gas will be needed to drive 650 miles. Assuming the car continues to use gas at the same rate, how many more gallons will be needed? Set up a rational equation and solve.

2. What is the formula you learned in Algebra I concerning distance, rate, and time? Write a rational equation solved for time. Set up a rational equation and use it to solve the following problem: Jerry walks 6 miles per hour and travels for 5 miles. How many minutes does he walk?

3. Sue and Bob are walking down an airport concourse at the same speed. Bob jumps on a 600 foot moving sidewalk that travels 3 feet per second and ends at the airplane door. While on the sidewalk, he continues to walk at the same rate as Sue until he reaches the end. He beats Sue by 180 seconds. (a) Using the formula in #2, write the rational expression for Sue’s time. (b) Write the rational expression for Bob’s time. (c) Since Bob’s time is 180 seconds less than Sue’s time, write the rational equation that equates their times. (d) Solve for the walking rate.

4. List the 6-step process for solving application problems developed in Unit 1.

5. Remember the Algebra I formula: Amount of work ($A$) = rate ($r$) times time ($t$). Rewrite the equation as the rational equation isolating: . Mary plants flowers at a rate of 200 seeds per hour. How many seeds has she planted in 2 hours? Write the rational equation and answer in a sentence.

6. If one whole job can be accomplished in $t$ units of time, then the rate of work is $r = \frac{1}{t}$. Harry and Melanie are working on Lake Pontchartrain clean-up detail. (a) Harry can clean up the trash in his area in 6 hours. Write an equation for Harry’s rate. (b) Melanie can do the same job in 4 hours. Write an equation for Melanie’s rate. (c) If they work together, how long will it take them to clean that area? Write a rational equation for the job and solve.
Application Problems

1. John’s car uses 18 gallons to travel 300 miles. He has 7 gallons of gas in the car and wants to know how much more gas will be needed to drive 650 miles. Assuming the car continues to use gas at the same rate, how many more gallons will be needed? Set up a rational equation and solve.

\[
\frac{18 \text{ gal.}}{300 \text{ mi.}} = \frac{(7 + x) \text{ gal.}}{650 \text{ mi.}}, \quad x = 32, \text{ John will need 32 more gallons to drive 650 miles.}
\]

2. What is the formula you learned in Algebra I concerning distance, rate, and time? \( d = rt \)

Write a rational equation solved for time. \( t = \frac{d}{r} \)

Set up a rational equation and use it to solve the following problem: Jerry walks 6 miles per hour and travels for 5 miles. How many minutes does he walk?

\( t = \frac{5 \text{ mi}}{6 \text{ mph}} \) or \( 5\frac{1}{6} \) of an hour, Jerry walks 50 minutes.

3. Sue and Bob are walking down an airport concourse at the same speed. Bob jumps on a 600 foot moving sidewalk that travels 3 feet per second and ends at the airplane door. While on the sidewalk, he continues to walk at the same rate as Sue until he reaches the end. He beats Sue by 180 seconds.

(a) Using the formula in #2, write the rational expression for Sue’s time.

\( \frac{600}{r} \)

(b) Write the rational expression for Bob’s time.

\( \frac{600}{r + 3} \)

(c) Since Bob’s time is 180 seconds less than Sue’s time, write the rational equation that equates their times.

\( \frac{600}{r} - 180 = \frac{600}{r + 3} \)

(d) Solve for the walking rate.

\( r = 2 \), Sue and Bob are walking at a rate of 2 feet per second.

4. List the 6-step process for solving application problems developed in Unit 1.

(1) Define the variables and the given information
(2) Determine what you are asked to find
(3) Write an equation
(4) Solve the equation
(5) Check
(6) Answer the question in a sentence, include units

5. Remember the Algebra I formula: Amount of work \( (A) = \) rate \( (r) \) times time \( (t) \). Rewrite the equation as the rational equation isolating \( r \): \( r = \frac{A}{t} \).

Mary plants flowers at a rate of 200 seeds per hour. How many seeds has she planted in 2 hours? Write the rational equation and answer in a sentence.

\( 200 = \frac{A}{2} \) Mary planted 400 seeds in 2 hours.

6. If one whole job can be accomplished in \( t \) units of time, then the rate of work is \( r = \frac{1}{t} \). Harry and Melanie are working on Lake Pontchartrain clean-up detail.

(a) Harry can clean up the trash in his area in 6 hours. Write an equation for Harry’s rate.

\( r = \frac{1}{6} \)

(b) Melanie can do the same job in 4 hours. Write an equation for Melanie’s rate.

\( r = \frac{1}{4} \)

(c) If they work together, how long will it take them to clean that area? Write a rational equation for the job and solve.

\( \frac{1}{6} + \frac{1}{4} = \frac{1}{t}, t = 2.4 \) It will take them 2.4 hours to clean the area if they work together.
Concept: Equation of a Circle  
Time Frame: 2 days

Common Core State Standard
Interpret the structure of expressions
A-SSE.2\textsuperscript{38} Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. In Algebra II, tasks are limited to polynomial, rational, or exponential expressions. Examples: see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. In the equation $x^2 + 2x + 1 + y^2 = 9$, see an opportunity to rewrite the first three terms as $(x+1)^2$, thus recognizing the equation of a circle with radius 3 and center $(-1, 0)$. See $(x^2 + 4)/(x^2 + 3)$ as $((x^2+3) + 1)/(x^2+3)$, thus recognizing an opportunity to write it as $1 + 1/(x^2 + 3)$. Can include the sum or difference of cubes, and factoring by grouping.

Objectives/Skills
- To write the equation of a circle given a center and radius with or without a graph
- Determine points on a circle
- Identify the center and radius given the equation of a circle

Examples:
Write the equation for a circle whose center is (2,10) and whose radius is 5.

Write the equation for a circle with endpoints of a diameter at (1,1) and (5,5).

What is an equation of circle $O$ shown in the graph below?

![Circle Diagram](attachment:image.png)

Resources
- Quick Interactive Quiz for writing equations of circles: [http://www.regentsprep.org/Regents/math/geometry/GCG6/PracCir.htm](http://www.regentsprep.org/Regents/math/geometry/GCG6/PracCir.htm)
- Interactive Quiz on Identifying the center and radius: [http://www.khanacademy.org/math/trigonometry/conics_precalc/circles-tutorial-precalc/e/equation_of_a_circle_1](http://www.khanacademy.org/math/trigonometry/conics_precalc/circles-tutorial-precalc/e/equation_of_a_circle_1)
- Notes and examples (good resource for absent students) Contains answers:
Module 1: Polynomial, Rational, and Radical Relationships

- Common Core

- HotMath - problems where answers can be revealed one step at a time: 
  http://hotmath.com/help/gt/genericalg2/section_12_2.html

- Worksheet – write equation from graph - 
### Module 1: Polynomial, Rational, and Radical Relationships

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<thead>
<tr>
<th>Concept: Systems of Equations</th>
<th>Time Frame: 2 days</th>
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<tr>
<td><strong>Common Core State Standard</strong></td>
<td></td>
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<tr>
<td>Solve systems of equations</td>
<td></td>
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<tr>
<td><strong>A-REI.6</strong> Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.</td>
<td></td>
</tr>
<tr>
<td><strong>A-REI.7</strong> Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.</td>
<td></td>
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<tr>
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<tr>
<td>Solve systems algebraically</td>
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<td>Solve systems graphically</td>
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<tr>
<td>Solve systems with two variables</td>
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<td>Solve linear with non-linear systems</td>
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<table>
<thead>
<tr>
<th>Examples:</th>
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<tbody>
<tr>
<td>Solve the following system:</td>
</tr>
</tbody>
</table>
| • $y = x^2 + 3x + 2$  
  $y = 2x + 3$ |
| • $y = -x - 3$  
  $x^2 + y^2 = 17$ |
| • $y = x^2 + 3x + 2$  
  $y = 2x + 3$ |
| • $x + 2y - z = 4$  
  $2x + y + z = -2$  
  $x + 2y + z = 2$ |

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Graphing Systems and solving systems with two variables (Henrico): <a href="http://teachers.henrico.k12.va.us/math/hcpsalgebra2/3-1.htm">http://teachers.henrico.k12.va.us/math/hcpsalgebra2/3-1.htm</a></td>
</tr>
<tr>
<td>Solving systems algebraically (two variables): <a href="http://teachers.henrico.k12.va.us/math/hcpsalgebra2/3-2.htm">http://teachers.henrico.k12.va.us/math/hcpsalgebra2/3-2.htm</a></td>
</tr>
<tr>
<td>Notes and examples for equations with three variables: <a href="http://tutorial.math.lamar.edu/Classes/Alg/SystemsThreeVrble.aspx">http://tutorial.math.lamar.edu/Classes/Alg/SystemsThreeVrble.aspx</a></td>
</tr>
<tr>
<td>Notes, examples and sample problems for systems with three variables: <a href="http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut50_systre">http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut50_systre</a></td>
</tr>
</tbody>
</table>
- Math Planet – example with embedded video: 

- Power Point on Systems in three variables: 

- Power Point on Systems with three variables (includes Geometric reasoning): 
  http://www.google.com/url?sa=t&rct=j&q=&esrc=s&frm=1&source=web&cd=3&ved=0CDQQFjAC&url=http%3A%2F%2Fteacherweb.com%2FKS%2FEll-SalineHighSchool%2FAmandaPhillimore%2F4.3-Linear-Systems-in-3-Variables.PPT&ei=eFMSUvqcBqji2QWKylGgCA&usg=AFQjCNG_Kqef7mO_FG9toTilSsKPJKA&surf=1&safe=active

- Henrico for Three variables:  http://teachers.henrico.k12.va.us/math/hcpsalgebra2/3-7.htm
Concept: Graphing Parabolas

Common Core State Standard

Translate between the geometric description and the equation for a conic section
G-GPE.2 Derive the equation of a parabola given a focus and directrix.

Objectives/Skills

- Write the equation of a parabola given the focus and directrix
- Find the y-intercept of a parabola
- Demonstrate points on the parabola are equidistant from the directrix

Examples

Identify the focus and directrix of the parabola given by \( y^2 = -4x \).

Identify the focus and directrix of the parabola given by \( x^2 = 12y \).

Write the standard form of the equation of the parabola with its vertex at (0, 0) and focus at (0, -4)

Write the standard form of the equation of the parabola with its vertex at (0, 0) and directrix \( y = 5 \)

Write the standard form of the equation of the parabola with its vertex at (0, 0) and directrix \( x = 2 \)

Strategies, Resources, Etc:

- Contains notes and picture examples: http://www.purplemath.com/modules/parabola.htm
- Contains a variety of worksheets for graphing parabolas: http://www.sanjuan.edu/webpages/mnarlesky/algebra.cfm?subpage=167597
- Matching activity with equation and graph: http://mathpl.us/docs/MatchParabolas.pdf
Concept: Congruent & Similar Parabolas

Estimated Days: 2 days

Common Core State Standard
Reason quantitatively and use units to solve problems.
N-Q.2 Define appropriate quantities for the purpose of descriptive modeling.

Analyze functions using different representations
F-IF-7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Translate between the geometric description and the equation for a conic section
G-GPE.2 Derive the equation of a parabola given a focus and directrix.

Objectives/Skills
- Determine if parabolas are congruent
- Prove parabolas congruent
- Determine if parabolas are similar

Examples
Identify and prove congruent parabolas.
1) $y = 2(x + 3)^2 - 8$
2) $y = (x + 1)^2 + 3$
3) $y = -(x + 3) - 5$
4) $y = -3(x - 2)^2 + 6$
5) $y = \frac{1}{2} (x + 2)^2 - 4$
6) $y = x^2 + 6x + 5$
7) $y = -\frac{1}{2}x^2 + x - \frac{3}{2}$
8) $y = -2x^2 + 12x - 9$
9) $y = 3x^2 + 2$
10) $y = -x^2 + \frac{1}{2}$
Answers:
1, 8, 9
2, 3, 6
4, 10
5, 7, 11

Strategies, Resources, Etc:
- Suggestion: Review congruent polygons, i.e. congruent trapezoids, to demonstrate how congruent figures map onto each other. Use the same idea to identify congruent parabolas. By doing several examples, students should realize that parabolas are congruent when the a is the same.
- Basic notes (most beneficial for teacher to review, not student): [http://www.ping.be/~ping1339/parabola.htm#Similar-parabolas](http://www.ping.be/~ping1339/parabola.htm#Similar-parabolas)
**Concept:** Solving Quadratic Equations using the Quadratic Formula

**Time Frame:** 4 days

**Common Core State Standard (GLE)**
A.REI.4: a. use the method of completing the square to transform any quadratic equation in x into an equation of the form \((x - p)^2 = q\) that has the same solutions. Derive the quadratic formula from this form. b. Solve quadratic equations by inspection (e.g., for \(x^2 = 49\)), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as \(a + bi\) for real numbers \(a\) and \(b\).

**Objectives/Skills**
- Solve quadratic equations using the quadratic formula
- Explain the complex solution result when the radicand is negative in the formula.
- Model real world situation

**Examples**
- \(x^2 - 12x = 28\)
- \(x^2 + 22x + 121 = 0\)
- \(2x^2 + 4x - 5 = 0\)
- \(x^2 - 4x = -13\)
- \(9x^2 - 12x + 4 = 0\)
- \(2x^2 + 16x + 33 = 0\)

**Instructional Strategies**

**NOTE:** AT THIS TIME, THE MODULE ADDRESSES ONLY REAL ROOTS. COMPLEX ROOTS ARE ADDRESSED LATER.
- Quadratic formula song example at [http://www.youtube.com/watch?v=79F2QxpjBz0](http://www.youtube.com/watch?v=79F2QxpjBz0)
- Quadratic Formula Lesson Plan [http://twt.borderlink.org/30667/5066/5066.html](http://twt.borderlink.org/30667/5066/5066.html)

**Resources**

**Comprehensive Curriculum:**
- Unit 5, Activity 4: The Quadratic Formula
- Unit 5, Activity 5: Use the Discriminant & the Graph to Determine the Nature of the Roots
- Unit 5, Activity 9: Solving Equations in Quadratic Form

**Textbook:**
- Section 6.5 The Quadratic Formula & The Discriminant (p. 313)
- Also see Section 5.9 Complex Numbers (p. 270)

**Websites:**
- [http://www.purplemath.com/modules/quadform.htm](http://www.purplemath.com/modules/quadform.htm)

**Enrichment:**
- [http://www.sosmath.com/algebra/quadrateeq/quadraformula/quadraformula.html](http://www.sosmath.com/algebra/quadrateeq/quadraformula/quadraformula.html) -- the equation derived
- [http://plus.maths.org/content/101-uses-quadratic-equation-part-ii](http://plus.maths.org/content/101-uses-quadratic-equation-part-ii) -- 101 uses of the quadratic formula (an article)
- [http://www.purplemath.com/modules/quadprob.htm](http://www.purplemath.com/modules/quadprob.htm) - word problems

**External Resources:**
- Algebra II: Structure and Method, book 2 by Houghton Mifflin – Chapter 10
- [http://www.youtube.com/watch?v=s8OJ2dAUyYl](http://www.youtube.com/watch?v=s8OJ2dAUyYl) -- video
- Algebra I, Chapter 10, section 4
- [http://teachers.henrico.k12.va.us/math/hcpsalgebra2/6-4.htm](http://teachers.henrico.k12.va.us/math/hcpsalgebra2/6-4.htm)
Activity 4: The Quadratic Formula

Materials List: paper, pencil, graphing calculator

Students will develop the quadratic formula and use it to solve quadratic equations.

Math Log Bellringer:
Solve the following quadratic equations using any method:
(1) \(x^2 - 25 = 0\)
(2) \(x^2 + 7 = 0\)
(3) \(x^2 + 4x = 12\)
(4) \(x^2 + 4x = 11\)
(5) Discuss the methods you used and why you chose that method.

Solutions: (1) \(x = 5, -5\), (2) \(x = \pm \sqrt{7}\), (3) \(x = -6, 2\), (4) \(x = -2 \pm \sqrt{15}\), (5) Answers will vary: factoring, isolating \(x^2\) and taking the square root of both sides, and completing the square.

Activity:

- Use the Bellringer to check for understanding of solving quadratic equations by all methods. Emphasize that Bellringer problem #4 must be solved by completing the square because it does not factor into rational numbers.

- Use the following process of completing the square to develop the quadratic formula.
\[
ax^2 + bx + c = 0
\]
\[
ax^2 + bx = -c
\]
\[
x^2 + \frac{b}{a}x = -\frac{c}{a}
\]
\[
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}
\]
\[
\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}
\]
\[
\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}
\]
\[
\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}
\]
\[
x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}
\]
\[
x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\]

- Use the quadratic formula to solve all four Bellringer problems.
• Use the math textbook for additional problems.
• Relating quadratic formula answers to graphing calculator zeroes: Have the students put \( y = x^2 + 4x - 7 \) in their calculators, find the zeroes, and then use the quadratic formula to find the zeroes. Use the calculator to find the decimal representation for the quadratic formula answers and compare the results. Discuss difference in exact and decimal approximation.

Activity 5: Using the Discriminant and the Graph to Determine the Nature of the Roots

Materials List: paper, pencil, graphing calculator

In this activity, students will examine the graphs of shifted quadratic functions, determine the types of roots and zeroes from the graph and from the discriminant, and describe the difference in a root and zero of a function.

Math Log Bellringer:
Find the roots of the following functions analytically.
(1) \( f(x) = x^2 + 4x - 5 \)
(2) \( f(x) = x^2 - 5 \)
(3) \( f(x) = x^2 - 4x + 4 \)
(4) \( f(x) = x^2 - 3x + 7 \)
(5) Graph the above functions on your calculator and describe the differences in the graphs, zeroes, and roots.

Solutions: (1) \( x = -5, 1 \), (2) \( x = \pm \sqrt{5} \), (3) \( x = 2 \), (4) \( x = \frac{3 \pm i\sqrt{19}}{2} \)

(5) #1 has two zeroes and two real rational roots,

#2 has two zeroes and two real irrational roots,

#3 has one zero and one real rational double root,

#4 has no zeroes and two complex (imaginary) roots

Activity:

• Use the Bellringer to check understanding of finding zeroes and relating them to the graph. Review the definition of double root from Unit 2 and what it looks like on a graph.

Module 1: Polynomial, Rational, and Radical Relationships
• Have students set up the Quadratic Formula for each of the equations in the Bellringer.
  o Solutions: (1) \( \frac{-4 \pm \sqrt{36}}{2} \), (2) \( \frac{0 \pm \sqrt{20}}{2} \), (3) \( \frac{4 \pm \sqrt{0}}{2} \), (4) \( \frac{3 \pm \sqrt{-19}}{2} \)
  o Have students determine from the set ups above what part of the formula determines if the roots are real or imaginary, rational or irrational, one, two or no roots.
  o Define \( b^2 - 4ac \) as the discriminant and have the students develop the rules concerning the nature of the solutions of the quadratic equation.
    1. If \( b^2 - 4ac = 0 \) \( \Rightarrow \) one zero and one real double root
    2. If \( b^2 - 4ac > 0 \) \( \Rightarrow \) two zeroes and two real roots
    3. If \( b^2 - 4ac < 0 \) \( \Rightarrow \) no zeroes and two imaginary roots
  o Emphasize the difference in the word root, which can be real or imaginary, and the word zero, which refers to an x-intercept of a graph.

• Assign problems from the textbook to practice predicting solutions using the discriminant.

• Application:
  Put students in pairs to determine if the following application problem has a solution using a discriminant: The length of the rectangle is twice the length of the side of the square and the width of the rectangle is 5 less than the side of the square. The area of a square is 40 more than the area of a rectangle. Find the length of the side of the square.
  (1) Draw pictures with the dimensions and set up the equation to compare areas. Use a discriminant to determine if this scenario is possible. Explain why your solution is possible or not.
  (2) Find a scenario that would make the solution possible, discuss, and solve.

  Solution:
  (1) \( s^2 = 2s(s - 5) + 40 \Rightarrow 0 = s^2 - 10s + 40 \). The discriminant = \(-60\) therefore a solution is not possible,
  (2) Answers will vary, but one scenario is an area of a square that is \(< 25\) more than the area of the rectangle.
Activity 9: Solving Equations in Quadratic Form

Materials List: paper, pencil

The students will examine equations that are not truly quadratic but in which they can use the same strategies to solve.

Math Log Bellringer:

Solve the following for \( t \): \( 2t^2 - 4t + 1 = 0 \) and discuss which method you used and why.

Solution: \( t = \frac{2 + \sqrt{2}}{2} \) and \( t = \frac{2 - \sqrt{2}}{2} \), I used the quadratic formula because I could not factor the equation.

Activity:

- Use the Bellringer to review the quadratic formula making sure to have students use the variable \( t \) in the quadratic formula and in the answer, then write the two answers separately. Substitute \( (s - 3) \) for \( t \) in the equation and ask them how to solve. Remind students to check the answers to prove that they are solutions and not extraneous roots.

Solution:

\[
\begin{align*}
2(s - 3)^2 - 4(s - 3) + 1 &= 0 \\
s - 3 &= \frac{2 + \sqrt{2}}{2} \text{ or } s - 3 = \frac{2 - \sqrt{2}}{2} \\
s &= \frac{2 + \sqrt{2}}{2} + 3 \text{ or } s = \frac{2 - \sqrt{2}}{2} + 3
\end{align*}
\]

Finding a common denominator: \( s = \frac{8 + \sqrt{2}}{2} \) and \( s = \frac{8 - \sqrt{2}}{2} \)

- Define quadratic form as any equation that can be written in the form \( at^2 + bt + c \) where \( t \) is any expression of a variable. Have students identify the expression that would be \( t \) in the following to make the equation quadratic form:

  1. \( x^4 + 7x^2 + 6 = 0 \)
  2. \( 2(y + 4)^2 + (y + 4) + 6 = 0 \)
  3. \( x - 3\sqrt{x} - 4 = 0 \)
  4. \( s^4 + 2s^2 = 0 \)

Solutions: (1) \( t = x^2 \), (2) \( t = \sqrt{x} \), (3) \( t = y + 4 \), (4) \( t = s^2 \)

- Have students work in pairs to solve the problems above making sure to check answers for extraneous roots.

Solution: (1) \( x = \left\{ \pm i, \pm i\sqrt{6} \right\} \), (2) \( y = \left\{ -\frac{5}{2}, -6 \right\} \), (3) \( x = 16 \), (4) \( s = \{0, \pm 3i\} \)

- Application: In a certain electrical circuit, the resistance of any \( R \), greater than 6 ohms, is found by solving the quadratic equation \( (R - 6)^2 = 4(R - 6) + 5 \). Show all of your work.

  1. Find \( R \) by solving the equation using quadratic form.
  2. Find \( R \) by first expanding the binomials and factoring.
  3. Find \( R \) by expanding the binomials then quadratic formula.
  4. Find \( R \) by graphing \( f(R) = (R - 6)^2 - 4(R - 6) - 5 \) and finding the zeroes.
  5. Discuss which of the above methods you like the best and why both solutions for \( R \) are not used.

Solution: \( R = 11 \) ohms, 5 ohms is not valid for the initial conditions

Module 1: Polynomial, Rational, and Radical Relationships
### Concept: Solve Quadratic Equations by Completing the Square

**Time Frame:** 4 days

**Common Core State Standard (GLE)**

A.REI.4:–

- **a.** Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

- **b.** Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers $a$ and $b$.

### Objectives/Skills

- Identify and factor perfect square trinomials (a=1 and a= factors other than 1)
- Solve quadratic equations by completing the square
- Model real world Situations

### Examples

- $x^2 + 8x - 20 = 0$
- $x^2 + 3x - 18 = 0$
- $2x^2 - 5x + 3 = 0$
- $3x^2 - 5x + 1 = 0$

### Instructional Strategies

- Use algebra tiles to complete the square:
  - [http://teachers.henrico.k12.va.us/math/hcpsalgebra2/6-3.htm](http://teachers.henrico.k12.va.us/math/hcpsalgebra2/6-3.htm)

### Resources

**Comprehensive Curriculum:**

- Unit 5, Activity 3: Completing the Square

**Textbook:**

- Section 6.4 Completing the Square (p. 306)

**Websites:**

- [http://www.purplemath.com/modules/sqrquad.htm](http://www.purplemath.com/modules/sqrquad.htm)
- [www.kutasoftware.com](http://www.kutasoftware.com)

**External Resources:**

- [http://www.youtube.com/watch?v=xGOQYTo9AKY](http://www.youtube.com/watch?v=xGOQYTo9AKY)

### Remediation:

- [http://www.mathwarehouse.com/quadratic/completing-the-square-math.php](http://www.mathwarehouse.com/quadratic/completing-the-square-math.php) -- can plug in coefficients and it will walk you through process
- [http://www.mathsisfun.com/algebra/completing-square.html](http://www.mathsisfun.com/algebra/completing-square.html) -- walk through for students

### Enrichment:

Activity 3: Completing the Square

Materials List: paper, pencil

In this activity, students will review solving quadratic equations by factoring and will learn to solve quadratic equations by completing the square.

Math Log Bellringer:
Solve the following for $x$:
(1) $x^2 - 8x + 7 = 0$
(2) $x^2 - 9 = 0$
(3) $x^2 = 16$
(4) $x^2 = -16$
(5) $(x - 4)^2 = 25$
(6) $(x - 2)^2 = -4$
(7) Discuss the difference in the way you solved #1 and #3

Solutions:
(1) $x = 7, 1$, (2) $x = 3, -3$, (3) $x = 4, -4$, (4) $x = 4i, -4i$, (5) $x = 9, -1$, (6) $x = 2i + 2, -2i + 2$, (7) To solve #1, I factored and used the Zero Property of Equations. To solve #3, I took the square root of both sides to get ±.

Activity:

- Use the Bellringer to:
  (1) Review the rules for factoring and the Zero Property of Equations for problems #1 and #2.
  (2) Review the rules for taking the square root of both sides in problems #3 and 4 with real and complex answers, reiterating the difference between the answer for $\sqrt{16}$ and the solution to the equation $x^2 = 16$. (The solution to $\sqrt{16} = 4$ is only the positive root, but the solutions to $x^2 = 16$ are ±4.)
  (3) Discuss the two methods that can be used to solve problem #5: (1) expand, isolate zero, and factor or (2) take the square root of both sides and isolate the variable.
  (4) Discuss whether both of these methods can be used to solve problem #6.

- Have students factor the expressions $x^2 + 6x + 9$ and $x^2 - 10x + 25$ to determine what properties of the middle term make these the square of a binomial (i.e. $(x \pm c)^2$). (Rule: If the leading coefficient is 1, and the middle coefficient is double the ±square root of the constant term, then it is a perfect square of a binomial (i.e. $6 = +2\sqrt{9}$ and $-10 = -2\sqrt{25}$). Have students check their conclusions by expanding $(x + d)^2 = x^2 + 2dx + d^2$ and $(x - d)^2 = x^2 - 2dx + d^2$. These are called perfect square trinomials.

- Have students find $c$ so the expressions $x^2 + 8x + c$ and $x^2 - 18x + c$ will be squares of binomials or perfect square trinomials. Name this process “completing the square” and have the students develop a set of steps to solve by this process.
  (1) Move all constants to the right side.
  (2) If the leading coefficient is not 1, factor out the leading coefficient and divide both sides by the leading coefficient.
  (3) Take $\frac{1}{2}$ the middle coefficient of $x$ and square it to find the constant, adding the same quantity.
to the both sides of the equation.
(4) Write the perfect square trinomial as a binomial squared.
(5) Take the square root of both sides making sure to get ±.
(6) Isolate $x$ for the two solutions.

• Guided Practice: Solve $3x^2 + 18x - 9 = 15$ by completing the square showing all the steps.
  Solution: Steps:
  1. $3x^2 + 18x = 24$
  2. $x^2 + 6x = 8$
  3. $x^2 + 6x + 9 = 8 + 9$
  4. $(x + 3)^2 = 17$
  5. $x + 3 = \pm \sqrt{17}$
  6. $x = -3 + \sqrt{17}$ or $x = -3 - \sqrt{17}$

• Assign problems from the textbook to practice solving quadratic equations by completing the square whose solutions are both real and complex.

• Application:
Put students in pairs to solve the following application problem:
(1) A farmer has 120 feet of fencing to fence in a dog yard next to the barn. He will use part of the barn wall as one side and wants the yard to have an area of 1000 square feet. What dimensions will the three sides of the yard be? (Draw a picture of the problem. Set up an equation to solve the problem by completing the square showing all the steps.)
(2) Suppose the farmer wants to enclose four sides with 120 feet of fencing. What are the dimensions to have an area of 1000 square feet? (Draw a picture of the problem. Set up an equation to solve the problem. Find the solution by completing the square showing all the steps.)
(3) Approximately how much fencing would be needed to enclose 1000 ft$^2$ on four sides? Discuss how you determined the answer.
  Solutions:
  (1) Perimeter: $w + w + \text{length} = 120 \implies \text{length} = 120 - 2w$
  Area: $(120 - 2w)w = 1000$
  $120w - 2w^2 = 1000$
  $-2(w^2 - 60w) = 1000$
  $w^2 - 60w = -500$
  $w^2 - 60w + 900 = -500 + 900$
  $(w - 30)^2 = 400$
  $w - 30 = \pm 20$
  $w = 50$ or $w = 10$, so there are two possible scenarios: (1) the three sides of the yard could be (1) 10, 10 and 100 ft or (2) 50, 50 and 20 feet
  (2) Perimeter: $2w + 2 \text{lengths} = 120 \implies \text{length} = 60 - 2w$
  Area: $w(60 - w) = 1000$
  $60w - w^2 = 1000$
  $w^2 - 60w = -1000$
  $(w^2 - 60w + 900) = -1000 + 900$
  $(w - 30)^2 = -10$
  There is not enough fencing to enclose 1000 ft$^2$.
  (4) I need to get a positive number when I complete the square so considering the equation $w^2 - bw + c = -1000 + c$, $c$ must be > 1000 therefore $\frac{b}{2} > \frac{1}{2} \implies b > \sqrt{1000} \approx 31.623 \implies b \approx 63.245$. Since $2b = \text{perimeter}$, you will need approximately 126.491 ft of fencing.
**Module 1: Polynomial, Rational, and Radical Relationships**

<table>
<thead>
<tr>
<th>Concept: Complex Numbers</th>
<th>Time Frame: 5 days</th>
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<tbody>
<tr>
<td><strong>Common Core State Standard</strong></td>
<td></td>
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<tr>
<td>Perform arithmetic operations with complex numbers.</td>
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<tr>
<td>N-CN.1 Know there is a complex number ( i ) such that ( i^2 = -1 ), and every complex number has the form ( a + bi ) with ( a ) and ( b ) real.</td>
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<tr>
<td>N-CN.2 Use the relation ( i^2 = -1 ) and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.</td>
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<tr>
<th>Objectives/Skills</th>
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<tr>
<td>• Add and subtract complex numbers</td>
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<tr>
<td>• Multiply and divide complex numbers</td>
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<tr>
<td>• Solve equations with imaginary solutions (this is continued from the unit on quadratics)</td>
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<tr>
<th>Examples:</th>
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| • **Simplify** \( 2i + 3i \).  
  \( 2i + 3i = (2 + 3)i = 5i \) |
| • **Simplify** \( 16i - 5i \).  
  \( 16i - 5i = (16 - 5)i = 11i \) |
| • **Multiply and simplify** \( (3i)(4i) \).  
  \( (3i)(4i) = (3\cdot4)(i\cdot i) = (12)(-1) = -12 \) |
| • **Multiply and simplify** \( (i)(2i)(-3i) \).  
  \( (i)(2i)(-3i) = (2\cdot3)(i\cdot i\cdot i) = (-6)(i^2 \cdot i) 
  = (-6)(-1\cdot i) = (-6)(-i) = 6i \) |
| • **Simplify** \( (2 + 3i)(1 - 6i) \).  
  \( (2 + 3i)(1 - 6i) = (2 + 1) + (3i - 6i) = 3 + (-3i) = 3 - 3i \) |
| • **Simplify** \( \frac{3}{2i} \).  
  \( \frac{3}{2i} = \frac{3}{2i} \cdot \frac{i}{i} = \frac{3i}{2(-1)} = \frac{3i}{-2} = -\frac{3i}{2} \) |

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<thead>
<tr>
<th>Instructional Strategies</th>
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<tbody>
<tr>
<td>• Complex Number Notebook: <a href="http://exchange.smarttech.com/details.html?id=70fe0af4-bc8d-4d14-bf93-5275aa03fb2d">http://exchange.smarttech.com/details.html?id=70fe0af4-bc8d-4d14-bf93-5275aa03fb2d</a></td>
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<td>• <a href="http://teachers.henrico.k12.va.us/math/hcpsalgebra2/5-10.htm">http://teachers.henrico.k12.va.us/math/hcpsalgebra2/5-10.htm</a></td>
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<th>Resources</th>
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<tr>
<td><strong>Comprehensive Curriculum:</strong></td>
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<tr>
<td>• Unit 4, Activity 6, 7, 8</td>
</tr>
<tr>
<td><strong>Textbook:</strong></td>
</tr>
<tr>
<td>• Section 5.9 – Complex Numbers, p. 270</td>
</tr>
<tr>
<td><strong>External Resources:</strong></td>
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<tr>
<td>• Showdown attached below</td>
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Activity 6: Imaginary Numbers
Materials List: paper, pencil, graphing calculator
In this activity, students will develop the concept of imaginary numbers and determine their place in the complex number system. They will simplify square root radicals whose radicands are negative and rationalize the denominator of fractions with imaginary numbers in the denominator.

Math Log Bellringer:
I. Graph the following without a graphing calculator and find the zeroes:
   (1) \( y = x^2 - 4 \)
   (2) \( y = x^2 - 8 \)
   (3) \( y = x^2 \)
   (4) \( y = x^2 - 1 \)
   (5) \( y = x^2 + 1 \)

II. Solve the following analytically and determine if the roots are rational or irrational:
   (6) \( x^2 - 4 = 0 \)
   (7) \( x^2 - 8 = 0 \)
   (8) \( x^2 = 0 \)
   (9) \( x^2 - 1 = 0 \)
   (10) \( x^2 + 1 = 0 \)

III. Explain the relationship between the problems in I and II above.
Solutions:
(1) \( x = \pm 2 \),
(2) \( x = \pm 2\sqrt{2} \),
(3) \( x = 0, \) double root,
(4) \( x = \pm 1 \),
(5) no zeroes,
(6) \( x = \pm 2, \) rational,
(7) \( x = \pm 2\sqrt{2}, \) irrational,
(8) \( x = 0, \) rational,
(9) \( x = \pm 1, \) rational,
(10) \( x = \sqrt{-1} \)
Activity:

- Use the Bellringer to review the definition of zeroes, the number of roots of a polynomial, and a double root. Determine that $\sqrt{-1}$ is the number needed to solve the equation: $x^2 + 1 = 0$. Define that number as the number $i$ in the set of Imaginary numbers which in union with the set of Real numbers makeup the set of Complex numbers. If $\sqrt{-1} = i$, then $i^2 = -1$ and for all positive real numbers $b$, $\sqrt{-b} = i\sqrt{b}$.

- Have students simplify $\sqrt[-7]{7}$, $\sqrt{-16}$, $\sqrt{-24}$
  
  **Solutions:** $\sqrt{-7} = i\sqrt{7}$, $\sqrt{-16} = 4i$, $\sqrt{-24} = 2i\sqrt{6}$

- Put students in pairs to determine the values of $i^2$, $i^3$, $i^4$, $i^5$, $i^6$, $i^7$, $i^8$, $i^9$, and have them write a rule that will help determine the answer to $i^{27}$, $i^{37}$, $i^{42}$, and $i^{20}$.

- Review the term, rationalize the denominator, and discuss how it applies to a problem in the form $\frac{4}{3i}$. Discuss how to use the rules of $i$ to rationalize this denominator. Since $i$ is an imaginary number, rationalizing the denominator means making the denominator a rational number; therefore, no $i$ can be in the denominator. Use the property that $i^2 = -1$, which is a rational number. **Solution:** $\frac{4i}{3i} = \frac{4i}{3i^2} = \frac{4i}{-3}$

- Return the students to pairs to rationalize the denominator of the following:
  
  1. $\frac{3\sqrt{5}}{\sqrt{-6}}$, 2. $\frac{6}{i^3}$, 3. $\frac{i^6}{i^{12}}$.

  **Solutions:** 1. $\frac{3\sqrt{5}}{\sqrt{-6}} = \frac{-i\sqrt{30}}{-2}$, 2. $\frac{6}{i^3} = 6i$, 3. $\frac{i^6}{i^{12}} = i$
Activity 7: Properties and Operations on Complex Numbers

Materials List: paper, pencil, Complex Number System BLM, overhead transparency or large chart paper for each pair of students

In this activity, students will develop the Complex number system and develop all operations on complex numbers including absolute value of a complex number.

Math Log Bellringer:
Distribute the Complex Number System BLM on which students should individually complete the word grid (view literacy strategy descriptions) and then compare their answers with a partner. Challenge students to find other sets of numbers and examples to add to the word grid to be discussed at a later date. (e.g., algebraic numbers, transcendental numbers, perfect numbers, prime numbers, composite numbers, and surds).

Activity:
• Use the Bellringer to define complex numbers \( \mathbb{C} \) as any number in the form of \( a + bi \) in which \( a \) and \( b \) are real numbers and \( i \) is \( \sqrt{-1} \). Redefine the set of Real numbers as numbers in the form \( a + bi \) where \( b = 0 \), and Imaginary numbers as numbers in the form \( a + bi \) where \( a = 0 \) and \( b \neq 0 \). Therefore, if the Complex number is \( a + bi \), then the real part is \( a \), and the imaginary part is \( b \). The complex conjugate is defined as \( a - bi \).

• Have the students refer to the Venn diagram they created in Activity 2 and add the set of Complex numbers and Imaginary numbers in the following manner.

and the list of properties are on the Complex Number System BLM.) When creating any new number system, certain mathematical terms must be defined. To review the meaning of these terms in the Real number system and to allow students to define them in the Complex number system, divide students into teams and assign each team an equal number of properties. Give each team a piece of chart paper or overhead transparency for each different property. Have them define what
they think the property is in words (verbally), and using $a + bi$ (symbolically), give a Complex number example without using the book. Have each member of the team present the property to the class, and let the class decide if the team should earn three points for that property. The team with the most points wins a bonus point (or candy, etc.).

- As the students present the properties to the class in the Complex Property Race above, have the students use split–page notetaking (view literacy strategy descriptions) to record the properties in their notebooks. The approach is modeled on the Complex Number System BLM with sample split-page notes from the properties. Explain the value of taking notes in this format by saying it logically organizes information and ideas from multiple sources; it helps separate big ideas from supporting details; it promotes active reading and listening; it allows inductive and deductive prompting for rehearsing and remembering the information.

- Assign more problems from the math textbook in which students have to add, subtract, multiply, and divide Complex numbers.
Do You Really Know the Difference?

State whether the following numbers are real (R) or imaginary (I) and discuss why.

(2) \( i \) \hspace{1cm} (8) the difference of an imaginary number and its conjugate
(3) \( i^2 \) \hspace{1cm} (9) the product of an imaginary number and its conjugate
(4) \( \sqrt{-9} \) \hspace{1cm} (10) the conjugate of an imaginary number
(5) \( \sqrt{-2\sqrt{-5}} \) \hspace{1cm} (11) the conjugate of a real number
(6) \( i^n \) if \( n \) is even \hspace{1cm} (12) the reciprocal of an imaginary number
(7) the sum of an imaginary number and its conjugate \hspace{1cm} (13) the additive inverse of an imaginary number
the product of an imaginary number and its conjugate \hspace{1cm} (14) the multiplicative identity of an imaginary number
the conjugate of an imaginary number \hspace{1cm} (15) the additive identity of an imaginary number

Answers:

(1) \( I \) This is the imaginary number equal to \( \sqrt{-1} \).
(2) \( R \) \( i^2 = 1 \) which is real.
(3) \( I \) \( \sqrt{-9} = 3i \) which is imaginary.
(4) \( R \) \( \sqrt{-2\sqrt{-5}} = (i\sqrt{2})(i\sqrt{5}) = i^2\sqrt{10} = -\sqrt{10} \) which is real.
(5) \( R \) If \( n \) is even then \( i^n \) will either be 1 or \(-1\) which are real.
(6) \( R \) \( (a + bi) + (a - bi) = 2a \) which is real.
(7) \( I \) \( (a + bi) - (a - bi) = 2bi \) which is imaginary.
(8) \( R \) \( (a + bi)(a - bi) = a^2 + b^2 \) which is real.
(9) \( I \) The conjugate of \( (0 + bi) \) is \( (0 - bi) \) which is imaginary.
(10) \( R \) The conjugate of \( (a + 0i) \) is \( (a - 0i) \) which is real.
(11) \( I \) The reciprocal of \( i \) is \( \frac{1}{i} \) which equals \(-i\) when you rationalize the denominator – imaginary.
(12) \( I \) The additive inverse of \( (0 + bi) \) is \( (0 - bi) \) which is imaginary.
(13) \( R \) The multiplicative identity of \( (0 + bi) \) is \( (1 + 0i) \) which is real.
(14) \( R \) The additive identity of \( (0 + bi) \) is \( (0 + 0i) \) which is real.
Activity 8: Finding Complex Roots of an Equation

Materials List: paper, pencil, graphing calculators

In this activity, students will find the complex roots of an equation and will reinforce the difference in root and zeroes using technology.

Math Log Bellringer:
Solve the following equations analytically and write all answers in \( a + bi \) form:

1. \( x^2 - 16 = 0 \)
2. \( x^2 + 16 = 0 \)
3. \( x^2 + 50 = 0 \)
4. \( (2x - 3)^2 - 18 = 0 \)
5. \( (3x - 2)^2 + 24 = 0 \)
6. \( x^3 - 28x = 0 \)
7. \( x^3 + 32x = 0 \)
Solutions:

(1) $\pm 4 + 0i$, (2) $0 \pm 4i$, (3) $0 \pm 5i\sqrt{2}$, (4) $\frac{3 + 3\sqrt{2}}{2} + 0i$, 

(5) $-\frac{2}{3} \pm \frac{2i\sqrt{6}}{3}$, (6) $0, \pm 2\sqrt{7} + 0i$, (7) $0, 0 \pm 4i\sqrt{2}$
Activity:

- Have students classify each of the answers of the Bellringer as real or imaginary.
- Have students graph each of the equations in the Bellringer in their graphing calculators and draw conclusions about (1) the number of roots, (2) types of roots, and (3) number of zeroes of a polynomial. Review the definitions of roots and zeroes: root ≡ the solution to a single variable equation which can be real or imaginary; zero ≡ the x value where y equals zero which is always real. Reiterate that the x– and y–axes on the graph are real numbers; therefore, a zero is an x–intercept.

Solutions:

(1) 2 real rational roots, 2 zeroes

(2) 2 imaginary roots, no zeroes

(3) 2 imaginary roots, no zeroes

(4) 2 real irrational roots and 2 zeroes

(5) 2 imaginary roots, no zeroes

(6) 3 roots, 2 irrational and 1 rational, 3 zeroes

(7) 3 roots, 1 real & rational, 2 imaginary, 1 zero

- Review solving polynomials by factoring using the Zero Property. Have the students predict the number of roots of $x^4 - 16 = 0$, solve it by factoring into $(x + 2)(x - 2)(x^2 + 4) = 0$ and applying the Zero Property, then predict the number of zeroes and end behavior of the graph of $y = x^4 - 16$. 

Module 1: Polynomial, Rational, and Radical Relationships

116 Common Core
### Concept: Simplifying Radicals

**Time Frame:** 1 days

**Common Core State Standard**

N-RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

### Objectives/Skills

- Simplify radicals and radical expressions
- Add, subtract, multiply, and divide radical expressions
- Rationalize the denominator
- Write expressions with rational exponents in radical form and vice versa
- Simplify expressions in exponential or radical form
- Review the concepts of simplifying \( n^{th} \) roots and solving equations of the form \( x^n = k \) in order to develop the properties of radicals and to simplify more complex radicals.

### Examples:

**Simplify** \( \sqrt{20r^{18} s^2 t^{21}} \)

Simplify by writing with no more than one radical: \( \sqrt[4]{4x} \cdot \sqrt[5]{x^3} \)

**Simplify:** \( \sqrt{9} + \sqrt{25} \)

**Simplify:** \( \sqrt{18} - 2\sqrt{27} + 3\sqrt{3} - 6\sqrt{8} \)

**Expand:** \( \sqrt{2} (3 + \sqrt{3}) \)

**Simplify:** \( \left( \sqrt[3]{3} + \sqrt[5]{5} \right) \left( \sqrt[3]{5} - \sqrt[6]{6} \right) \)

**Simplify:** \( \sqrt{\frac{25}{64}} \)

**Simplify:** \( \frac{\sqrt{3}}{1 + \sqrt{7}} \)

**Simplify:** \( 2 - \sqrt{7} \)

### Instructional Strategies

- Radical Thinking attached
- [http://teachers.henrico.k12.va.us/math/hcpsalgebra2/5-6.htm](http://teachers.henrico.k12.va.us/math/hcpsalgebra2/5-6.htm)
- [http://teachers.henrico.k12.va.us/math/hcpsalgebra2/5-7.htm](http://teachers.henrico.k12.va.us/math/hcpsalgebra2/5-7.htm)

### Resources

**Comprehensive Curriculum:**
- Unit 4, Activity 1, 2, 3

**Textbook:**
- Section 5.5, Roots of Real Numbers,

**Remediation:**
Module 1: Polynomial, Rational, and Radical Relationships

- Section 5.6, Radical Expressions, p. 250
- Section 5.7, Rational Exponents, p. 257

Websites:
- http://www.ixl.com/math/algebra-1

Videos:

Activity 1: Roots and Radicals

Materials List: paper, pencil, graphing calculator, Math Log Bellringer BLM

In this activity, the students will review the concepts of simplifying $n^{th}$ roots and solving equations of the form $x^n = k$ in order to develop the properties of radicals and to simplify more complex radicals.

Emphasis in this lesson is on the new concept that $\sqrt{x^2} = |x|$.

Math Log Bellringer:

Graph on the graphing calculator and find the points of intersection:

1. $y_1 = x^2$ and $y_2 = 9$
2. $y_1 = x^2$ and $y_2 = -9$
3. $y_1 = x^2$ and $y_2 = 0$
4. Discuss the number of points of intersection each set of equations has.

Solutions:

1. $(±3, 9)$
2. empty set
3. $(0, 0)$

(4) There are 2 solutions to #1, no solutions to #2, 1 solution to #3 (a double root)

Activity:

- Overview of the Math Log Bellringers:
  - As in previous units, each in-class activity in Unit 4 is started with an activity called a Math Log Bellringer that either reviews past concepts to check for understanding (i.e., reflective thinking about what was learned in previous classes or previous courses) or sets the stage for an upcoming concept (i.e., predictive thinking for that day’s lesson).
  - A math log is a form of a learning log (view literacy strategy descriptions) that students keep in order to record ideas, questions, reactions, and new understandings. Documenting ideas in a log about content being studied forces students to “put into words” what they know or do not know. This process offers a reflection of understanding that can lead to further study and alternative learning paths. It combines writing and reading with content learning. The Math Log Bellringers will include mathematics done symbolically, graphically, and verbally.
  - Since Bellringers are relatively short, blackline masters have not been created for each of them. Write them on the board before students enter class, paste them into an enlarged Word* document or PowerPoint® slide, and project using a TV or digital projector, or print and display using a document or overhead projector. A sample enlarged Math Log Bellringer Word* document has been included in the blackline masters. This sample is the Math Log Bellringer for this activity.

Module 1: Polynomial, Rational, and Radical Relationships

119  Common Core
Have the students write the Math Log Bellringers in their notebooks, preceding the upcoming lesson during beginning—of—class record keeping, and then circulate to give individual attention to students who are weak in that area.

- Use the Bellringer to generate a discussion about the number of answers for \(x^2 = 9, \ x^2 = -9, \ x^2 = 0\). Review the definition of root as the solution to an equation in one variable. Ask how this definition relates to the use of the word square root.

- Have the students define the terms index, radical, radicand. Have the students enter \(y_1 = \sqrt{x}\) and trace to \(x = 9\) in their calculators. There is one answer, 3, as opposed to the solution of #1 in the Bellringer, which has two answers, ±3. Discuss the definition of principal square root as the positive square root.

- Ask the students to solve the following:

  1. \(\sqrt{6^2}\)
  2. \(\sqrt[3]{(-6)^2}\)
  3. \(\sqrt[3]{2^3}\)
  4. \(\sqrt[3]{(-2)^3}\)

  Solutions: (1) 6, (2) 6, (3) 2, (4) -2

- Add the following problems to the list above (5) \(\sqrt{x^2}\) and (6) \(\sqrt[3]{x^3}\). Discuss solutions. The students will usually answer “\(x\)” as the solution to both. Have them enter \(y = \sqrt{x^2}\) and \(y = \sqrt[3]{x^3}\) in their graphing calculators and identify the graphs as \(y = |x|\) and \(y = x\). Review the piecewise function for \(|x|\) and how it relates to \(\sqrt{x^2}\).

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]

- A very important concept for future mathematical study is the progression of the following solution.

  - Explain and lead a class discussion about its importance.

  1. \(x^2 = 9\)
  2. \(\sqrt{x^2} = \sqrt{9}\)
  3. \(|x| = 3\)
  4. \(x = \pm 3\)

- Have the students solve some radical problems with different indices and develop the rules for \(n^{th}\) root of \(b^n\) where \(n\) is even and \(n\) is odd.

  1. If \(n\) is even, then \(\sqrt[n]{b^n} = |b| = \begin{cases} 
  b & \text{if } b \geq 0 \\
  -b & \text{if } b < 0 
\end{cases}\)

  2. If \(n\) is odd, then \(\sqrt[n]{b^n} = b\)
• Have students determine how the \( n^{\text{th}} \) root rule can be applied to expressions with multiple radicands such as \( \sqrt[2]{81x^4} = \sqrt[2]{(9x^2)^2} = 9x^2 \), discussing why absolute value is not needed in this situation. \( (9x^2 \text{ is always positive.}) \) Work and discuss \( \sqrt[3]{64x^{15}} = \sqrt[3]{(4x^5)^3} = 4x^5 \).

• Review the rules for solving absolute value equations and have students apply \( n^{\text{th}} \) root rule to solve \( \sqrt{(x+3)^2} = 12 \)

\[
\text{Solution: } \sqrt{(x+3)^2} = 12 \Rightarrow |x+3| = 9 \Rightarrow x = 6 \text{ or } -15
\]
Application:
Meteorologists have determined that the duration of a storm is dependent on the diameter of the storm. The function \( f(d) = 0.07(\sqrt{d})^3 \) defines the relationship where \( d \) is the diameter of the storm in miles and \( f(d) \) is the duration in hours. How long will a storm last if the diameter of the storm is 9 miles? Write the answer in function notation with the answer in decimals and write the answer in a sentence in hours and minutes.

Solution:
\( f(9) = 1.890 \), The storm will last approximately 1 hour and 53 minutes.
Graph on the graphing calculator and find the points of intersection:

(1) \( y_1 = x^2 \) and \( y_2 = 9 \)

(2) \( y_1 = x^2 \) and \( y_2 = -9 \)

(3) \( y_1 = x^2 \) and \( y_2 = 0 \)

(4) Discuss the number of points of intersection each set of equations has.
Activity 2: Multiplying and Dividing Radicals

Materials List: paper, pencil, graphing calculator, Sets of Numbers BLM, Multiplying & Dividing Radicals BLM

In this activity, the students will review the product and quotient rules for radicals addressed in Algebra I. They will use them to multiply, divide, and simplify radicals with variables in the radicand.

Math Log Bellringer:
Simplify showing the steps used:

1. \( \sqrt{50} \)
2. \( \sqrt[3]{-40} \)
3. \( \sqrt[3]{8} \)
4. \( \sqrt[3]{2} \)
5. Write the rules symbolically and verbally for multiplying and dividing radicals.

Solutions:

1. \( \sqrt{50} = \sqrt{(25)(2)} = \sqrt{5^2 \cdot 2} = 5\sqrt{2} \)
2. \( \sqrt[3]{-40} = \sqrt[3]{(-8)(5)} = \sqrt[3]{(-2)^3 \cdot 5} = -2\sqrt[3]{5} \)
3. \( \sqrt[3]{\frac{8}{9}} = \sqrt[3]{\frac{(2^3)}{3^2}} = \frac{2\sqrt[3]{2}}{3} \)
4. \( \frac{2}{\sqrt[3]{5}} = \frac{2\sqrt[3]{5}}{\sqrt[3]{5^2}} = \frac{2\sqrt[3]{5}}{5} \)
5. If \( \sqrt[n]{a} \) and \( \sqrt[n]{b} \) are real numbers and \( n \) is a natural number, then
   - \( \sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b} \). The radical of a product equals the product of two radicals.
   - \( \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \). The radical of a quotient is the quotient of two radicals, \( b \neq 0 \).

Activity:

- Have students put both the Bellringer problems and answers in the calculators on the home screen to check for equivalency. This can be done by getting decimal representations or using the TEST feature of the calculator: Enter \( \sqrt{50} = 5\sqrt{2} \) (The “=” sign is found under 2ND [TEST], (above MATH). If the calculator returns a “1”, then the statement is true; if it returns a “0”, then the statement is false.)
• Discuss why the product rule does not apply in the following situation: \( \sqrt{(-4)(-9)} \neq -4\sqrt{-9} \) (Have students enter \( \sqrt{-4} \) in the calculator. They will get this error message, \( \sqrt{-4} \). The rule says, “If \( \sqrt{a} \) and \( \sqrt{b} \) are real numbers.” These nonreal numbers will be discussed in Activity 7.)

• Reviewing Sets of Numbers:
  - In this activity, the students will use the Venn diagram which is a form of graphic organizer (view literacy strategy descriptions) to review the mathematical relationships between different sets of numbers. Later in this unit in Activity 7, the students will add the sets of imaginary numbers and complex numbers to their Venn diagrams.
  - Distribute the Sets of Numbers BLM and give the students an opportunity to work in pairs to complete the diagram and to write as much information as they can remember about the sets of numbers.
  - Put a large Venn diagram on the board and have the students volunteer their answers and correct their worksheets.
  - Develop a definition for rational numbers and apply the definition to the Bellringer problems.

• Multiplying and Dividing Radicals Worksheet:
  - This is a guided worksheet for the students to review simplifying radicals of numbers and rationalizing the denominator. They will then apply this previous knowledge to multiply and divide radicals with variables.
  - Distribute the Multiplying & Dividing Radicals BLM. Have students work in pairs to answer #1, 2, and 3. Lead a full class discussion of these concepts to develop the answer to #4.
  - Work the first problem in #5 with the class, and then give the students an opportunity to complete the worksheet. When everyone is finished, allow student volunteers to explain their processes on the board.
Reviewing Sets of Numbers

Fill in the following sets of numbers in the Venn diagram: natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers.

1. Write the symbol for the set and list its elements in set notation:
   - natural numbers: ____________ What is another name for natural numbers? ____________
   - whole numbers: __________________________
   - integers: ________________________________

2. Define rational numbers. What is its symbol and why? Give some examples. ________________
   ________________
   ________________

3. Are your Bellringers rational or irrational? Why? ________________
   ________________
Reviewing Sets of Numbers

Fill in the following sets of numbers in the Venn diagram: natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers.

1. Write the symbol for the set and list its elements in set notation:

   natural numbers: \( N = \{1, 2, 3, \ldots\} \)  
   What is another name for natural numbers? **Counting**

   whole numbers: \( W = \{0, 1, 2, 3, \ldots\} \)

   integers: \( \mathbb{I} \) or \( \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \)

2. Define rational numbers. What is its symbol and why? Give some examples. **Any number in the form \( \frac{p}{q} \) where \( p \) and \( q \) are integers, \( q \neq 0 \). The symbol is \( \mathbb{Q} \) for quotient. Ex. All repeating and terminating decimals and fractions of integers. \( 7, 7.5, 7.6666\ldots, \frac{1}{2}, -\frac{1}{3} \)**

3. Are your Bellringer problems rational or irrational? Why? **Irrational because they cannot be expressed as fractions of integers. Their decimal representations do not repeat or terminate.**
1. Can the product of two irrational numbers be a rational number? Give an example.

2. What does “rationalizing the denominator” mean and why do we rationalize the denominator?

3. Rationalize the following denominators and simplify:
   (1) \( \frac{1}{\sqrt{5}} \)  
   (2) \( \frac{1}{\sqrt{5}} \)  
   (3) \( \frac{1}{\sqrt{8}} \)

4. List what should be checked to make sure a radical is in simplest form:
   a.  
   b.  
   c.  

5. Simplify the following expressions applying rules to radicals with variables in the radicand.
   (1) \( \sqrt{72}x^3y^4 \)  
   (2) \( \sqrt{80}s^4t^6 \)  
   (3) \( \sqrt{2xy} \cdot \sqrt{6x^3y} \)  
   (4) \( 5\sqrt{4x^2y^5} \cdot 7\sqrt{2x^3y} \)  
   (5) \( \frac{\sqrt{162x^6}}{\sqrt{10x^7}} \)  
   (6) \( \frac{\sqrt{2s^2}}{\sqrt{18s^3}} \)

**Application**

The time in seconds, \( t(L) \), for one complete swing of a pendulum is dependent upon the length of the pendulum in feet, \( L \), and gravity which is 32 ft/sec\(^2\) on earth. It is modeled by the function

\[
t(L) = 2\pi \sqrt{\frac{L}{32}}.
\]

Find the time for one complete swing of a 4-foot pendulum. Express the exact simplified answer in function notation and express the answer in a sentence rounding to the nearest tenth of a second.
Multiplying and Dividing Radicals

1. Can the product of two irrational numbers be a rational number? Give an example.
   
   Yes, \( \sqrt{2} \sqrt{32} = 8 \)

2. What does “rationalizing the denominator” mean and why do we rationalize the denominator?
   
   Rationalizing the denominator means making sure that the number in the denominator is a rational number and not an irrational number with a radical. We rationalize denominators because we do not want to divide by a nonrepeating, nonterminating decimal.

3. Rationalize the following denominators and simplify:
   
   (1) \( \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \)
   
   (2) \( \frac{1}{\sqrt{25}} = \frac{\sqrt{25}}{5} \)
   
   (3) \( \frac{1}{\sqrt{8}} = \frac{\sqrt{2}}{4} \)

4. List what should be checked to make sure a radical is in simplest form:

   a. The radicand contains no exponent greater than or equal to the index
   
   b. The radicand contains no fractions
   
   c. The denominator contains no radicals

5. Simplify the following expressions applying rules to radicals with variables in the radicand.

   (1) \( \sqrt{72x^3y^4} = 6y^2|x|\sqrt{2x} \)
   
   (2) \( \sqrt{80s^4t^6} = 2st^2\sqrt{10s} \)
   
   (3) \( \sqrt{2xy} \cdot \sqrt{6x^2y} = 2x^2|y|\sqrt{y} \)
   
   (4) \( 5\sqrt{4x^2y^3} \cdot 7\sqrt{2x^2y} = 70xy^2\sqrt{x} \)
   
   (5) \( \frac{\sqrt{162x^6}}{\sqrt{10x^2}} = \frac{9\sqrt{5x}}{5x} \)
   
   (6) \( \frac{\sqrt[3]{2s^2}}{\sqrt[3]{18s^3}} = \frac{\sqrt[3]{3s^2}}{3s} \)
Application

The time in seconds, \( t(L) \), for one complete swing of a pendulum is dependent upon the length of the pendulum in feet, \( L \), and gravity which is \( 32 \text{ ft/sec}^2 \) on earth. It is modeled by the function \( t(L) = 2\pi \sqrt{\frac{L}{32}} \). Find the time for one complete swing of a 4-foot pendulum. Express the exact simplified answer in function notation and express the answer in a sentence rounding to the nearest tenth of a second. \( t(L) = \frac{\pi \sqrt{L}}{2} \). One complete swing of a 4-foot pendulum takes approximately 2.2 seconds.
Activity 3: Adding and Subtracting Radicals

Materials List: paper, pencil

In this activity, the students will review the sum and difference rules for radicals addressed in Algebra I and use them to add, subtract, and simplify radicals with variables in the radicand.

Math Log Bellringer:
Simplify
(1) $6x^2 + 4y - x + 5x^2 - 7y + 9x$
(2) $6\sqrt{2} + 4\sqrt{3} - \frac{3}{2} + 5\sqrt{2} - 7\sqrt{3} + 9\sqrt{2}$
(3) $(3x + 5)(7x - 9)$
(4) $(3\sqrt{2} + 5)(7\sqrt{2} - 9)$
(5) How do the rules of polynomials in #1 compare to the rules of radicals in #2?

Solutions:
(1) $11x^2 + 8x - 3y$, (2) $11\sqrt{2} + 8\sqrt{3} - 3\sqrt{3} + 8x - 45$, (4) $21\left(\sqrt{2}\right)^2 + 8\sqrt{2} - 45 = 42 + 8\sqrt{2} - 45 = 8\sqrt{2} - 3$
(5) Add the coefficients of like radicals.

Activity:
- Use the Bellringer to compare addition and multiplication of polynomials to addition and multiplication of radicals and to show how the distributive property is involved.
- Have students simplify the following to review a 9th Grade GLE: $6\sqrt{18} + 4\sqrt{8} - 3\sqrt{72}$ . Have students define “like radicals” as expressions that have the same index and same radicand, and then have students develop the rules for adding and subtracting radicals.
- Put students in pairs to simplify the following radicals:

  (1) $4\sqrt{18}x - \sqrt{72}x + \sqrt{50}x$
  (2) $\sqrt[3]{64xy^2} + \frac{1}{2}7x^4y^5$
  (3) $(2\sqrt{a} - 3\sqrt{b})(4\sqrt{a} + 7\sqrt{b})$
  (4) $(x + \sqrt{5})^2$
  (5) $(x + \sqrt{3})(x - \sqrt{3})$

Solutions:
(1) $11\sqrt{2}x$
(2) $(4 + 3xy)\sqrt[3]{xy^2}$
(3) $8a + 2\sqrt{ab} - 21b$
(4) $x^2 + 2x\sqrt{5} + 5$, (5) $x^2 - 3$
• Use problem #5 above to define conjugate and have students determine how to rationalize the denominator of \( \frac{1}{\sqrt{2} + \sqrt{5}} \).
**Module 1: Polynomial, Rational, and Radical Relationships**

### Concept: Solve Equations Containing Radicals

**Common Core State Standard (GLE)**

Understand solving equations as a process of reasoning and explain the reasoning

**A-REI.2** Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

### Objectives/Skills

- Solving equations containing radicals
- Solve word problems using real world situations
- Use technology to graph simple functions that involve radical expressions; determine domain, range, x- and y-intercepts
- Solve equations that involve radical expressions analytically as well as using technology, and apply the equations to real-world applications

### Examples

\[
a \sqrt[3]{3} + 2 = 2a \sqrt[3]{3} + 7
\]

\[
\sqrt{3d} + 1 = 4
\]

\[
\sqrt{x} - 8 = 0
\]

\[
1 + x\sqrt{2} = 0
\]

\[
\sqrt[3]{y} + 1 = 2
\]

\[
\sqrt{x} - 2 = \sqrt{3} + \sqrt{x}
\]

### Instructional Strategies

- Solving Radical Equations by Graphing p. 268 Glencoe textbook
- Radical and Exponential Notebook: [http://exchange.smarttech.com/details.html?id=751b16f1-caa1-4fc8-81a6-bcb113b26b53](http://exchange.smarttech.com/details.html?id=751b16f1-caa1-4fc8-81a6-bcb113b26b53)
- [http://teachers.henrico.k12.va.us/math/hcpsalgebra2/5-8.htm](http://teachers.henrico.k12.va.us/math/hcpsalgebra2/5-8.htm)

### Resources

#### Comprehensive Curriculum:
- Unit 4 Activity 4 and 5

#### Textbook:
- Section 5.8 Rational Equations p. 263
- Section 7.9 – Square Root Functions and Inequalities (Functions only)

#### Websites:
- [http://www.ixl.com/math/algebra-1](http://www.ixl.com/math/algebra-1)
- [http://www.purplemath.com/modules/solverad.htm](http://www.purplemath.com/modules/solverad.htm)

#### Remediation:
<table>
<thead>
<tr>
<th>Videos:</th>
</tr>
</thead>
</table>
Activity 4: Graphing the Radical Function

Materials List: paper, pencil, graphing calculator, Graphing Radical Functions Discovery Worksheet BLM

In this activity, the students will use technology to graph simple functions that involve radical expressions in preparation for solving equations involving radical expressions analytically. They will determine domain, range, and x- and y-intercepts.

Math Log Bellringer:

1. Set the window: $x: [-1, 10], y: [-5, 5]$. Graph $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$ on the graphing calculator and state the domain, range, and x- and y-intercepts.

2. From the graph screen find $f(8)$ and $g(8)$ and round three decimal places. Are these answers rational or irrational numbers?

3. Graph $y = 2$ and use the intersection feature of the calculator to solve $-2 = \sqrt{x}$ and $-2 = \sqrt[3]{x}$.

Solutions:

1. Domain: $x \geq 0$, Range $y \geq 0$, (0, 0)

2. $f(8) = 2.828$ irrational, $g(8) = 2$, rational

3. No solution to $-2 = \sqrt{x}$

   One solution $(-8, -2)$

Activity:

- Use the Bellringer to review graphing calculator skills.
- Graphing Radical Functions Discovery Worksheet:
  - Divide the students in pairs and distribute Graphing Radical Functions Discovery Worksheet BLM. On this worksheet the students will learn how to graph translations of square root and cube root functions.
  - When students have finished the worksheet, make sure they have come to the correct conclusions in #10 and 11.
- Assign similar homework from the math textbook.
Module 1: Polynomial, Rational, and Radical Relationships

### Radical Graph Translations

<table>
<thead>
<tr>
<th>Equation</th>
<th>Sketch</th>
<th>Domain</th>
<th>Range</th>
<th>x-intercept</th>
<th>y-intercept</th>
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</thead>
<tbody>
<tr>
<td>$y = \sqrt{x} + 3$</td>
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</tr>
<tr>
<td>$y = \sqrt{x} - 3$</td>
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</tr>
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<tr>
<td>$y = \sqrt{x - 3} + 5$</td>
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</tbody>
</table>

(10) What is the difference in the graph when a constant is added outside of the radical, $f(x) + k$, or inside of the radical, $f(x + k)$?

(11) What is the difference in the domains and ranges of $f(x) = \sqrt{x}$ and $g(x) = \sqrt[5]{x}$? Why is the domain of one of the functions restricted and the other not?
<table>
<thead>
<tr>
<th>Equation</th>
<th>Sketch</th>
<th>Domain</th>
<th>Range</th>
<th>x-intercept</th>
<th>y-intercept</th>
</tr>
</thead>
<tbody>
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<td>( y \geq 3 )</td>
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</tr>
<tr>
<td>2 ( y = \sqrt{x} - 3 )</td>
<td><img src="image2" alt="Graph" /></td>
<td>( x \geq 0 )</td>
<td>( y \geq -3 )</td>
<td>((9, 0))</td>
<td>((0, -3))</td>
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<tr>
<td>3 ( y = \sqrt[3]{x} + 2 )</td>
<td><img src="image3" alt="Graph" /></td>
<td>all reals</td>
<td>all reals</td>
<td>((-8, 0))</td>
<td>((0, 2))</td>
</tr>
<tr>
<td>4 ( y = \sqrt[3]{x} - 2 )</td>
<td><img src="image4" alt="Graph" /></td>
<td>all reals</td>
<td>all reals</td>
<td>((8, 0))</td>
<td>((0, -2))</td>
</tr>
<tr>
<td>5 ( y = \sqrt{x - 4} )</td>
<td><img src="image5" alt="Graph" /></td>
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<td>( y \geq 0 )</td>
<td>((4, 0))</td>
<td>none</td>
</tr>
<tr>
<td>6 ( y = \sqrt{x + 4} )</td>
<td><img src="image6" alt="Graph" /></td>
<td>( x \geq -4 )</td>
<td>( y \geq 0 )</td>
<td>((-4, 0))</td>
<td>((0, 2))</td>
</tr>
<tr>
<td>7 ( y = \sqrt[3]{x + 5} )</td>
<td><img src="image7" alt="Graph" /></td>
<td>all reals</td>
<td>all reals</td>
<td>((-5, 0))</td>
<td>((0, \sqrt[3]{5}))</td>
</tr>
<tr>
<td>8 ( y = \sqrt[3]{x - 5} )</td>
<td><img src="image8" alt="Graph" /></td>
<td>all reals</td>
<td>all reals</td>
<td>((5, 0))</td>
<td>((0, \sqrt[3]{-5}))</td>
</tr>
<tr>
<td>9 ( y = \sqrt{x - 3} + 5 )</td>
<td><img src="image9" alt="Graph" /></td>
<td>( x \geq 3 )</td>
<td>( y \geq 5 )</td>
<td>none</td>
<td>none</td>
</tr>
</tbody>
</table>
(10) What is the difference in the graph when a constant is added outside of the radical, \( f(x) + k \), or inside of the radical, \( f(x + k) \)? Outside the radical changes the vertical shift, + up and – down. A constant inside the radical, changes the horizontal shift, + left and n right.

(11) What is the difference in the domains and ranges of \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt[3]{x} \)? Why is the domain of one of the functions restricted and the other not? Even index radicals have a restricted domain \( x \geq 0 \) and therefore a resulting restricted range \( y \geq 0 \). The domain and range of odd index radicals are both all reals. You cannot take an even index radical of a negative number.
Activity 5: Solving Equations with Radical Expressions

Materials List: paper, pencil, graphing calculator

In this activity, students will solve equations that involve radical expressions analytically as well as using technology, and apply them to real-world applications.

Math Log Bellringer:

Use the graphing calculator to graph the two functions and find the points of intersection in order to solve:

(1) Graph $y_1 = \sqrt[3]{3x-2}$ and $y_2 = 4$ to solve $\sqrt[3]{3x-2} = 4$

(2) Graph $y_1 = \sqrt[3]{3x-2}$ and $y_2 = -4$ to solve $\sqrt[3]{3x-2} = -4$

(3) Explain the difference.

Solutions:

(1) $x = 6$

(2) no solution,

(3) A radical is never negative therefore there is no solution.

Activity:

• Ask the students to solve the following mentally, and have them discuss their thought processes:

  (1) $\sqrt{x} - 4 = 0$
  (2) $\sqrt[3]{x} = 2$
  (3) $\sqrt{x} = -5$

  Solutions: (1) $x = 16$, (2) $x = 8$, (3) empty set

• Define and discuss extraneous roots: extra roots that are not true solutions of the original radical equation. Use the discussion to generate steps to solve equations containing variables under radicals:
  1. Isolate the radical
  2. Raise both sides of the equation to a power that is the same as the index of the radical
  3. Solve for $x$
  4. Check

• Continuing from the previous three problems, have students solve the following analytically and discuss:

  (4) $\sqrt[3]{3x-2} = 4$
  (5) $\sqrt[3]{3x-2} = -4$

  (6) How are the problems above related to the graphs in the Bellringer?

  Solutions:
  (4) $x = 6$
  (5) no solution, $x = 6$ is an extraneous root
  (6) same

• Continuing from the above problems, have students solve the following analytically and graphically:

  (7) $\sqrt[3]{x-3} + 5 = x$. (Review the process of solving polynomials by factoring and using the zero property.)
Solution: \( x = 7 \)

(8) Graph both sides of the equation (i.e., \( y_1 = \sqrt{x-3} + 5 \) and \( y_2 = x \)) and explain why \( x = 7 \) is a solution and \( x = 4 \) is not.

Solution: The graphs intersect only once, so 4 is an extraneous root.

- Solve and check analytically and graphically:
  
  (9) \( \sqrt{3x+2} - \sqrt{2x+7} = 0 \)
  (10) \( \sqrt{x-5} - \sqrt{x} = 2 \)

Solutions:

(9) \( x = 5 \),

(10) no solution

- Application:

The length of the diagonal of a box is given by \( d = \sqrt{L^2 + W^2 + H^2} \). What is the length, \( L \), of the box if the height, \( H \), is 4 feet, the width, \( W \), is 5 feet and the diagonal, \( d \), is 9 feet? Express the answer in a sentence in feet and inches rounding to the nearest inch.

Solution: The length of the box is approximately 6 feet, 4 inches.